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MTH203-Course Portfolio-Spring 2021

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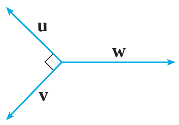
Table of contents

0.0.1 Problems Solutions on vectors

8. If the vectors in the figure satisfy $|\mathbf{u}| = |\mathbf{v}| = 1$ and $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$, what is $|\mathbf{w}|$?



9–14 Find a vector $\mathbf{a}$ with representation given by the directed line segment $\overrightarrow{AB}$. Draw $\overrightarrow{AB}$ and the equivalent representation starting at the origin.

9. $A(-1, 1)$, $B(3, 2)$

10. $A(-4, -1)$, $B(1, 2)$

11. $A(-1, 3)$, $B(2, 2)$

12. $A(2, 1)$, $B(0, 6)$

13. $A(0, 3, 1)$, $B(2, 3, -1)$

14. $A(4, 0, -2)$, $B(4, 2, 1)$

15–18 Find the sum of the given vectors and illustrate geometrically.

15. $\langle -1, 4 \rangle$, $\langle 6, -2 \rangle$

16. $\langle 3, -1 \rangle$, $\langle -1, 5 \rangle$

17. $\langle 3, 0, 1 \rangle$, $\langle 0, 8, 0 \rangle$

18. $\langle 1, 3, -2 \rangle$, $\langle 0, 0, 6 \rangle$

19–22 Find $\mathbf{a} + \mathbf{b}$, $2\mathbf{a} + 3\mathbf{b}$, $|\mathbf{a}|$, and $|\mathbf{a} - \mathbf{b}|$.

19. $\mathbf{a} = \langle 5, -12 \rangle$, $\mathbf{b} = \langle -3, -6 \rangle$

20. $\mathbf{a} = 4\,\mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{i} - 2\,\mathbf{j}$

21. $\mathbf{a} = \mathbf{i} + 2\,\mathbf{j} - 3\,\mathbf{k}$, $\mathbf{b} = -2\,\mathbf{i} - \mathbf{j} + 5\,\mathbf{k}$

22. $\mathbf{a} = 2\,\mathbf{i} - 4\,\mathbf{j} + 4\,\mathbf{k}$, $\mathbf{b} = 2\,\mathbf{j} - \mathbf{k}$

23–25 Find a unit vector that has the same direction as the given vector.

23. $-3\,\mathbf{i} + 7\,\mathbf{j}$

24. $\langle -4, 2, 4 \rangle$

25. $8\,\mathbf{i} - \mathbf{j} + 4\,\mathbf{k}$

26. Find a vector that has the same direction as $\langle -2, 4, 2 \rangle$ but has length 6.

27–28 What is the angle between the given vector and the positive direction of the x-axis?

27. $\mathbf{i} + \sqrt{3}\,\mathbf{j}$

28. $8\,\mathbf{i} + 6\,\mathbf{j}$

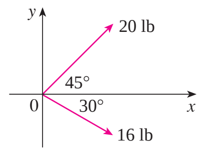
29. If $\mathbf{v}$ lies in the first quadrant and makes an angle $\pi/3$ with the positive x-axis and $|\mathbf{v}| = 4$, find $\mathbf{v}$ in component form.

30. If a child pulls a sled through the snow on a level path with a force of 50 N exerted at an angle of 38° above the horizontal, find the horizontal and vertical components of the force.

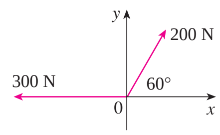
31. A quarterback throws a football with angle of elevation 40° and speed 60 ft/s. Find the horizontal and vertical components of the velocity vector.

32–33 Find the magnitude of the resultant force and the angle it makes with the positive x-axis.

32.



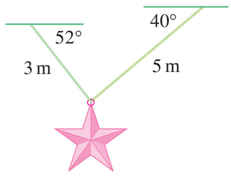
33.



34. The magnitude of a velocity vector is called *speed.* Suppose that a wind is blowing from the direction N45°W at a speed of 50 km/h. (This means that the direction from which the wind blows is 45° west of the northerly direction.) A pilot is steering a plane in the direction N60°E at an airspeed (speed in still air) of 250 km/h. The *true course,* or *track,* of the plane is the direction of the resultant of the velocity vectors of the plane and the wind. The *ground speed* of the plane is the magnitude of the resultant. Find the true course and the ground speed of the plane.

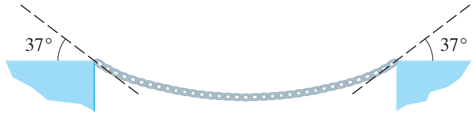
35. A woman walks due west on the deck of a ship at 3 mi/h. The ship is moving north at a speed of 22 mi/h. Find the speed and direction of the woman relative to the surface of the water.

36. Ropes 3 m and 5 m in length are fastened to a holiday decoration that is suspended over a town square. The decoration has a mass of 5 kg. The ropes, fastened at different heights, make angles of 52° and 40° with the horizontal. Find the tension in each wire and the magnitude of each tension.



37. A clothesline is tied between two poles, 8 m apart. The line is quite taut and has negligible sag. When a wet shirt with a mass of 0.8 kg is hung at the middle of the line, the midpoint is pulled down 8 cm. Find the tension in each half of the clothesline.

38. The tension $\mathbf{T}$ at each end of the chain has magnitude 25 N (see the figure). What is the weight of the chain?



39. A boatman wants to cross a canal that is 3 km wide and wants to land at a point 2 km upstream from his starting point. The current in the canal flows at 3.5 km/h and the speed of his boat is 13 km/h.
(a) In what direction should he steer?
(b) How long will the trip take?

20. $\mathbf{a}+\mathbf{b}=(4\,\mathbf{i}+\mathbf{j})+(\mathbf{i}-2\,\mathbf{j})=5\,\mathbf{i}-\mathbf{j}$

$2\mathbf{a}+3\mathbf{b}=2\,(4\,\mathbf{i}+\mathbf{j})+3\,(\mathbf{i}-2\,\mathbf{j})=8\,\mathbf{i}+2\,\mathbf{j}+3\,\mathbf{i}-6\,\mathbf{j}=11\,\mathbf{i}-4\,\mathbf{j}$

$|\mathbf{a}|=\sqrt{4^2+1^2}=\sqrt{17}$

$|\mathbf{a}-\mathbf{b}|=|(4\,\mathbf{i}+\mathbf{j})-(\mathbf{i}-2\,\mathbf{j})|=|3\,\mathbf{i}+3\,\mathbf{j}|=\sqrt{3^2+3^2}=\sqrt{18}=3\sqrt{2}$

21. $\mathbf{a}+\mathbf{b}=(\mathbf{i}+2\,\mathbf{j}-3\,\mathbf{k})+(-2\,\mathbf{i}-\mathbf{j}+5\,\mathbf{k})=-\,\mathbf{i}+\mathbf{j}+2\mathbf{k}$

$2\mathbf{a}+3\mathbf{b}=2\,(\mathbf{i}+2\,\mathbf{j}-3\,\mathbf{k})+3\,(-2\,\mathbf{i}-\mathbf{j}+5\,\mathbf{k})=2\,\mathbf{i}+4\,\mathbf{j}-6\,\mathbf{k}-6\,\mathbf{i}-3\,\mathbf{j}+15\,\mathbf{k}=-\,4\,\mathbf{i}+\mathbf{j}+9\mathbf{k}$

$|\mathbf{a}|=\sqrt{1^2+2^2+(-3)^2}=\sqrt{14}$

$|\mathbf{a}-\mathbf{b}|=|(\mathbf{i}+2\,\mathbf{j}-3\,\mathbf{k})-(-2\,\mathbf{i}-\mathbf{j}+5\,\mathbf{k})|=|3\,\mathbf{i}+3\,\mathbf{j}-8\,\mathbf{k}|=\sqrt{3^2+3^2+(-8)^2}=\sqrt{82}$

22. $\mathbf{a}+\mathbf{b}=(2\,\mathbf{i}-4\,\mathbf{j}+4\,\mathbf{k})+(2\,\mathbf{j}-\mathbf{k})=2\,\mathbf{i}-2\,\mathbf{j}+3\,\mathbf{k}$

$2\mathbf{a}+3\mathbf{b}=2\,(2\,\mathbf{i}-4\,\mathbf{j}+4\,\mathbf{k})+3\,(2\,\mathbf{j}-\mathbf{k})=4\,\mathbf{i}-8\,\mathbf{j}+8\,\mathbf{k}+6\mathbf{j}-3\,\mathbf{k}=4\,\mathbf{i}-2\,\mathbf{j}+5\,\mathbf{k}$

$|\mathbf{a}|=\sqrt{2^2+(-4)^2+4^2}=\sqrt{36}=6$

$|\mathbf{a}-\mathbf{b}|=|(2\,\mathbf{i}-4\,\mathbf{j}+4\,\mathbf{k})-(2\,\mathbf{j}-\mathbf{k})|=|2\,\mathbf{i}-6\,\mathbf{j}+5\,\mathbf{k}|=\sqrt{2^2+(-6)^2+5^2}=\sqrt{65}$

23. The vector $-3\,\mathbf{i}+7\,\mathbf{j}$ has length $|-3\,\mathbf{i}+7\,\mathbf{j}|=\sqrt{(-3)^2+7^2}=\sqrt{58}$, so by Equation 4 the unit vector with the same direction is $\dfrac{1}{\sqrt{58}}\,(-3\,\mathbf{i}+7\,\mathbf{j})=-\dfrac{3}{\sqrt{58}}\,\mathbf{i}+\dfrac{7}{\sqrt{58}}\,\mathbf{j}$.

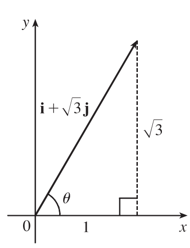
24. $|\langle -4,2,4\rangle|=\sqrt{(-4)^2+2^2+4^2}=\sqrt{36}=6$, so $\mathbf{u}=\frac{1}{6}\langle -4,2,4\rangle=\left\langle -\frac{2}{3},\frac{1}{3},\frac{2}{3}\right\rangle$.

25. The vector $8\,\mathbf{i}-\mathbf{j}+4\,\mathbf{k}$ has length $|8\,\mathbf{i}-\mathbf{j}+4\,\mathbf{k}|=\sqrt{8^2+(-1)^2+4^2}=\sqrt{81}=9$, so by Equation 4 the unit vector with the same direction is $\frac{1}{9}(8\,\mathbf{i}-\mathbf{j}+4\,\mathbf{k})=\frac{8}{9}\,\mathbf{i}-\frac{1}{9}\,\mathbf{j}+\frac{4}{9}\,\mathbf{k}$.

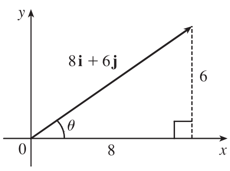
26. $|\langle -2,4,2\rangle|=\sqrt{(-2)^2+4^2+2^2}=\sqrt{24}=2\sqrt{6}$, so a unit vector in the direction of $\langle -2,4,2\rangle$ is $\mathbf{u}=\dfrac{1}{2\sqrt{6}}\,\langle -2,4,2\rangle$.

A vector in the same direction but with length 6 is $6\mathbf{u}=6\cdot\dfrac{1}{2\sqrt{6}}\,\langle -2,4,2\rangle=\left\langle -\dfrac{6}{\sqrt{6}},\dfrac{12}{\sqrt{6}},\dfrac{6}{\sqrt{6}}\right\rangle$ or $\langle -\sqrt{6},2\sqrt{6},\sqrt{6}\,\rangle$.

27.

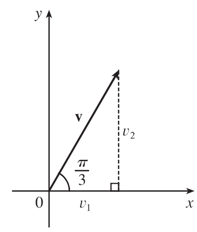


From the figure, we see that $\tan\theta=\dfrac{\sqrt{3}}{1}=\sqrt{3}\quad\Rightarrow\quad\theta=60^\circ$.

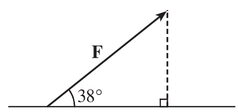
28. 

From the figure we see that $\tan\theta = \frac{6}{8} = \frac{3}{4}$, so $\theta = \tan^{-1}\left(\frac{3}{4}\right) \approx 36.9^\circ$.

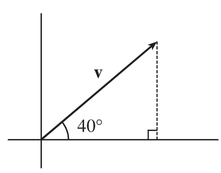
29. From the figure, we see that the x-component of $\mathbf{v}$ is $v_1 = |\mathbf{v}|\cos(\pi/3) = 4 \cdot \frac{1}{2} = 2$ and the y-component is $v_2 = |\mathbf{v}|\sin(\pi/3) = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}$. Thus $\mathbf{v} = \langle v_1, v_2 \rangle = \langle 2, 2\sqrt{3}\,\rangle$.



30. From the figure, we see that the horizontal component of the force $\mathbf{F}$ is $|\mathbf{F}|\cos 38^\circ = 50\cos 38^\circ \approx 39.4$ N, and the vertical component is $|\mathbf{F}|\sin 38^\circ = 50\sin 38^\circ \approx 30.8$ N.



31. The velocity vector $\mathbf{v}$ makes an angle of 40° with the horizontal and has magnitude equal to the speed at which the football was thrown. From the figure, we see that the horizontal component of $\mathbf{v}$ is $|\mathbf{v}|\cos 40^\circ = 60\cos 40^\circ \approx 45.96$ ft/s and the vertical component is $|\mathbf{v}|\sin 40^\circ = 60\sin 40^\circ \approx 38.57$ ft/s.



32. The given force vectors can be expressed in terms of their horizontal and vertical components as $20\cos 45^\circ\,\mathbf{i} + 20\sin 45^\circ\,\mathbf{j} = 10\sqrt{2}\,\mathbf{i} + 10\sqrt{2}\,\mathbf{j}$ and $16\cos 30^\circ\,\mathbf{i} - 16\sin 30^\circ\,\mathbf{j} = 8\sqrt{3}\,\mathbf{i} - 8\,\mathbf{j}$. The resultant force $\mathbf{F}$ is the sum of these two vectors: $\mathbf{F} = \left(10\sqrt{2} + 8\sqrt{3}\right)\mathbf{i} + \left(10\sqrt{2} - 8\right)\mathbf{j} \approx 28.00\,\mathbf{i} + 6.14\,\mathbf{j}$. Then we have $|\mathbf{F}| \approx \sqrt{(28.00)^2 + (6.14)^2} \approx 28.7$ lb and, letting θ be the angle $\mathbf{F}$ makes with the positive x-axis,

$$\tan\theta = \frac{10\sqrt{2} - 8}{10\sqrt{2} + 8\sqrt{3}} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{10\sqrt{2} - 8}{10\sqrt{2} + 8\sqrt{3}}\right) \approx 12.4^\circ.$$

33. The given force vectors can be expressed in terms of their horizontal and vertical components as $-300\,\mathbf{i}$ and $200\cos 60^\circ\,\mathbf{i} + 200\sin 60^\circ\,\mathbf{j} = 200\left(\frac{1}{2}\right)\,\mathbf{i} + 200\left(\frac{\sqrt{3}}{2}\right)\,\mathbf{j} = 100\,\mathbf{i} + 100\sqrt{3}\,\mathbf{j}$. The resultant force $\mathbf{F}$ is the sum of these two vectors: $\mathbf{F} = (-300 + 100)\,\mathbf{i} + \left(0 + 100\sqrt{3}\,\right)\mathbf{j} = -200\,\mathbf{i} + 100\sqrt{3}\,\mathbf{j}$. Then we have $|\mathbf{F}| \approx \sqrt{(-200)^2 + \left(100\sqrt{3}\,\right)^2} = \sqrt{70{,}000} = 100\sqrt{7} \approx 264.6$ N. Let θ be the angle $\mathbf{F}$ makes with the

0.0.2 Problems with solutions on Lines and planes

12.5 Exercises

1. Determine whether each statement is true or false.
   (a) Two lines parallel to a third line are parallel.
   (b) Two lines perpendicular to a third line are parallel.
   (c) Two planes parallel to a third plane are parallel.
   (d) Two planes perpendicular to a third plane are parallel.
   (e) Two lines parallel to a plane are parallel.
   (f) Two lines perpendicular to a plane are parallel.
   (g) Two planes parallel to a line are parallel.
   (h) Two planes perpendicular to a line are parallel.
   (i) Two planes either intersect or are parallel.
   (j) Two lines either intersect or are parallel.
   (k) A plane and a line either intersect or are parallel.

2–5 Find a vector equation and parametric equations for the line.

2. The line through the point $(6, -5, 2)$ and parallel to the vector $\langle 1, 3, -\frac{2}{3} \rangle$
3. The line through the point $(2, 2.4, 3.5)$ and parallel to the vector $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
4. The line through the point $(0, 14, -10)$ and parallel to the line $x = -1 + 2t$, $y = 6 - 3t$, $z = 3 + 9t$
5. The line through the point $(1, 0, 6)$ and perpendicular to the plane $x + 3y + z = 5$

6–12 Find parametric equations and symmetric equations for the line.

6. The line through the origin and the point $(4, 3, -1)$
7. The line through the points $\left(0, \frac{1}{2}, 1\right)$ and $(2, 1, -3)$
8. The line through the points $(1.0, 2.4, 4.6)$ and $(2.6, 1.2, 0.3)$
9. The line through the points $(-8, 1, 4)$ and $(3, -2, 4)$
10. The line through $(2, 1, 0)$ and perpendicular to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \mathbf{k}$
11. The line through $(1, -1, 1)$ and parallel to the line $x + 2 = \frac{1}{2}y = z - 3$
12. The line of intersection of the planes $x + 2y + 3z = 1$ and $x - y + z = 1$

13. Is the line through $(-4, -6, 1)$ and $(-2, 0, -3)$ parallel to the line through $(10, 18, 4)$ and $(5, 3, 14)$?
14. Is the line through $(-2, 4, 0)$ and $(1, 1, 1)$ perpendicular to the line through $(2, 3, 4)$ and $(3, -1, -8)$?
15. (a) Find symmetric equations for the line that passes through the point $(1, -5, 6)$ and is parallel to the vector $\langle -1, 2, -3 \rangle$.
    (b) Find the points in which the required line in part (a) intersects the coordinate planes.
16. (a) Find parametric equations for the line through $(2, 4, 6)$ that is perpendicular to the plane $x - y + 3z = 7$.
    (b) In what points does this line intersect the coordinate planes?
17. Find a vector equation for the line segment from $(2, -1, 4)$ to $(4, 6, 1)$.
18. Find parametric equations for the line segment from $(10, 3, 1)$ to $(5, 6, -3)$.

19–22 Determine whether the lines L_1 and L_2 are parallel, skew, or intersecting. If they intersect, find the point of intersection.

19. L_1: $x = 3 + 2t, \quad y = 4 - t, \quad z = 1 + 3t$
    L_2: $x = 1 + 4s, \quad y = 3 - 2s, \quad z = 4 + 5s$
20. L_1: $x = 5 - 12t, \quad y = 3 + 9t, \quad z = 1 - 3t$
    L_2: $x = 3 + 8s, \quad y = -6s, \quad z = 7 + 2s$
21. L_1: $\dfrac{x-2}{1} = \dfrac{y-3}{-2} = \dfrac{z-1}{-3}$
    L_2: $\dfrac{x-3}{1} = \dfrac{y+4}{3} = \dfrac{z-2}{-7}$
22. L_1: $\dfrac{x}{1} = \dfrac{y-1}{-1} = \dfrac{z-2}{3}$
    L_2: $\dfrac{x-2}{2} = \dfrac{y-3}{-2} = \dfrac{z}{7}$

23–40 Find an equation of the plane.

23. The plane through the origin and perpendicular to the vector $\langle 1, -2, 5 \rangle$
24. The plane through the point $(5, 3, 5)$ and with normal vector $2\mathbf{i} + \mathbf{j} - \mathbf{k}$
25. The plane through the point $\left(-1, \frac{1}{2}, 3\right)$ and with normal vector $\mathbf{i} + 4\mathbf{j} + \mathbf{k}$
26. The plane through the point $(2, 0, 1)$ and perpendicular to the line $x = 3t$, $y = 2 - t$, $z = 3 + 4t$
27. The plane through the point $(1, -1, -1)$ and parallel to the plane $5x - y - z = 6$
28. The plane through the point $(2, 4, 6)$ and parallel to the plane $z = x + y$
29. The plane through the point $\left(1, \frac{1}{2}, \frac{1}{3}\right)$ and parallel to the plane $x + y + z = 0$
30. The plane that contains the line $x = 1 + t$, $y = 2 - t$, $z = 4 - 3t$ and is parallel to the plane $5x + 2y + z = 1$
31. The plane through the points $(0, 1, 1)$, $(1, 0, 1)$, and $(1, 1, 0)$
32. The plane through the origin and the points $(2, -4, 6)$ and $(5, 1, 3)$

1. Homework Hints available at stewartcalculus.com

33. The plane through the points $(3, -1, 2)$, $(8, 2, 4)$, and $(-1, -2, -3)$

34. The plane that passes through the point $(1, 2, 3)$ and contains the line $x = 3t$, $y = 1 + t$, $z = 2 - t$

35. The plane that passes through the point $(6, 0, -2)$ and contains the line $x = 4 - 2t$, $y = 3 + 5t$, $z = 7 + 4t$

36. The plane that passes through the point $(1, -1, 1)$ and contains the line with symmetric equations $x = 2y = 3z$

37. The plane that passes through the point $(-1, 2, 1)$ and contains the line of intersection of the planes $x + y - z = 2$ and $2x - y + 3z = 1$

38. The plane that passes through the points $(0, -2, 5)$ and $(-1, 3, 1)$ and is perpendicular to the plane $2z = 5x + 4y$

39. The plane that passes through the point $(1, 5, 1)$ and is perpendicular to the planes $2x + y - 2z = 2$ and $x + 3z = 4$

40. The plane that passes through the line of intersection of the planes $x - z = 1$ and $y + 2z = 3$ and is perpendicular to the plane $x + y - 2z = 1$

41–44 Use intercepts to help sketch the plane.

41. $2x + 5y + z = 10$

42. $3x + y + 2z = 6$

43. $6x - 3y + 4z = 6$

44. $6x + 5y - 3z = 15$

45–47 Find the point at which the line intersects the given plane.

45. $x = 3 - t,\ y = 2 + t,\ z = 5t;\quad x - y + 2z = 9$

46. $x = 1 + 2t,\ y = 4t,\ z = 2 - 3t;\quad x + 2y - z + 1 = 0$

47. $x = y - 1 = 2z;\quad 4x - y + 3z = 8$

48. Where does the line through $(1, 0, 1)$ and $(4, -2, 2)$ intersect the plane $x + y + z = 6$?

49. Find direction numbers for the line of intersection of the planes $x + y + z = 1$ and $x + z = 0$.

50. Find the cosine of the angle between the planes $x + y + z = 0$ and $x + 2y + 3z = 1$.

51–56 Determine whether the planes are parallel, perpendicular, or neither. If neither, find the angle between them.

51. $x + 4y - 3z = 1,\quad -3x + 6y + 7z = 0$

52. $2z = 4y - x,\quad 3x - 12y + 6z = 1$

53. $x + y + z = 1,\quad x - y + z = 1$

54. $2x - 3y + 4z = 5,\quad x + 6y + 4z = 3$

55. $x = 4y - 2z,\quad 8y = 1 + 2x + 4z$

56. $x + 2y + 2z = 1,\quad 2x - y + 2z = 1$

57–58 (a) Find parametric equations for the line of intersection of the planes and (b) find the angle between the planes.

57. $x + y + z = 1,\quad x + 2y + 2z = 1$

58. $3x - 2y + z = 1,\quad 2x + y - 3z = 3$

59–60 Find symmetric equations for the line of intersection of the planes.

59. $5x - 2y - 2z = 1,\quad 4x + y + z = 6$

60. $z = 2x - y - 5,\quad z = 4x + 3y - 5$

61. Find an equation for the plane consisting of all points that are equidistant from the points $(1, 0, -2)$ and $(3, 4, 0)$.

62. Find an equation for the plane consisting of all points that are equidistant from the points $(2, 5, 5)$ and $(-6, 3, 1)$.

63. Find an equation of the plane with x-intercept a, y-intercept b, and z-intercept c.

64. (a) Find the point at which the given lines intersect:

$$\mathbf{r} = \langle 1, 1, 0 \rangle + t\langle 1, -1, 2 \rangle$$
$$\mathbf{r} = \langle 2, 0, 2 \rangle + s\langle -1, 1, 0 \rangle$$

(b) Find an equation of the plane that contains these lines.

65. Find parametric equations for the line through the point $(0, 1, 2)$ that is parallel to the plane $x + y + z = 2$ and perpendicular to the line $x = 1 + t$, $y = 1 - t$, $z = 2t$.

66. Find parametric equations for the line through the point $(0, 1, 2)$ that is perpendicular to the line $x = 1 + t$, $y = 1 - t$, $z = 2t$ and intersects this line.

67. Which of the following four planes are parallel? Are any of them identical?

P_1: $3x + 6y - 3z = 6$ $\qquad$ P_2: $4x - 12y + 8z = 5$

P_3: $9y = 1 + 3x + 6z$ $\qquad$ P_4: $z = x + 2y - 2$

68. Which of the following four lines are parallel? Are any of them identical?

L_1: $x = 1 + 6t,\quad y = 1 - 3t,\quad z = 12t + 5$

L_2: $x = 1 + 2t,\quad y = t,\quad z = 1 + 4t$

L_3: $2x - 2 = 4 - 4y = z + 1$

L_4: $\mathbf{r} = \langle 3, 1, 5 \rangle + t\langle 4, 2, 8 \rangle$

69–70 Use the formula in Exercise 45 in Section 12.4 to find the distance from the point to the given line.

69. $(4, 1, -2);\quad x = 1 + t,\ y = 3 - 2t,\ z = 4 - 3t$

70. $(0, 1, 3);\quad x = 2t,\ y = 6 - 2t,\ z = 3 + t$

71–72 Find the distance from the point to the given plane.

71. $(1, -2, 4)$, $3x + 2y + 6z = 5$

72. $(-6, 3, 5)$, $x - 2y - 4z = 8$

73–74 Find the distance between the given parallel planes.

73. $2x - 3y + z = 4$, $4x - 6y + 2z = 3$

74. $6z = 4y - 2x$, $9z = 1 - 3x + 6y$

75. Show that the distance between the parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is

$$D = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

76. Find equations of the planes that are parallel to the plane $x + 2y - 2z = 1$ and two units away from it.

77. Show that the lines with symmetric equations $x = y = z$ and $x + 1 = y/2 = z/3$ are skew, and find the distance between these lines.

78. Find the distance between the skew lines with parametric equations $x = 1 + t$, $y = 1 + 6t$, $z = 2t$, and $x = 1 + 2s$, $y = 5 + 15s$, $z = -2 + 6s$.

79. Let L_1 be the line through the origin and the point $(2, 0, -1)$. Let L_2 be the line through the points $(1, -1, 1)$ and $(4, 1, 3)$. Find the distance between L_1 and L_2.

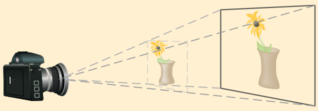
80. Let L_1 be the line through the points $(1, 2, 6)$ and $(2, 4, 8)$. Let L_2 be the line of intersection of the planes π_1 and π_2, where π_1 is the plane $x - y + 2z + 1 = 0$ and π_2 is the plane through the points $(3, 2, -1)$, $(0, 0, 1)$, and $(1, 2, 1)$. Calculate the distance between L_1 and L_2.

81. If a, b, and c are not all 0, show that the equation $ax + by + cz + d = 0$ represents a plane and $\langle a, b, c\rangle$ is a normal vector to the plane.
Hint: Suppose $a \neq 0$ and rewrite the equation in the form

$$a\left(x + \frac{d}{a}\right) + b(y - 0) + c(z - 0) = 0$$

82. Give a geometric description of each family of planes.
(a) $x + y + z = c$ (b) $x + y + cz = 1$
(c) $y \cos\theta + z \sin\theta = 1$

LABORATORY PROJECT PUTTING 3D IN PERSPECTIVE



Computer graphics programmers face the same challenge as the great painters of the past: how to represent a three-dimensional scene as a flat image on a two-dimensional plane (a screen or a canvas). To create the illusion of perspective, in which closer objects appear larger than those farther away, three-dimensional objects in the computer's memory are projected onto a rectangular screen window from a viewpoint where the eye, or camera, is located. The viewing volume—the portion of space that will be visible—is the region contained by the four planes that pass through the viewpoint and an edge of the screen window. If objects in the scene extend beyond these four planes, they must be truncated before pixel data are sent to the screen. These planes are therefore called *clipping planes*.

1. Suppose the screen is represented by a rectangle in the yz-plane with vertices $(0, \pm 400, 0)$ and $(0, \pm 400, 600)$, and the camera is placed at $(1000, 0, 0)$. A line L in the scene passes through the points $(230, -285, 102)$ and $(860, 105, 264)$. At what points should L be clipped by the clipping planes?
2. If the clipped line segment is projected on the screen window, identify the resulting line segment.
3. Use parametric equations to plot the edges of the screen window, the clipped line segment, and its projection on the screen window. Then add sight lines connecting the viewpoint to each end of the clipped segments to verify that the projection is correct.
4. A rectangle with vertices $(621, -147, 206)$, $(563, 31, 242)$, $(657, -111, 86)$, and $(599, 67, 122)$ is added to the scene. The line L intersects this rectangle. To make the rectangle appear opaque, a programmer can use *hidden line rendering*, which removes portions of objects that are behind other objects. Identify the portion of L that should be removed.

12.5 Equations of Lines and Planes

1. (a) True; each of the first two lines has a direction vector parallel to the direction vector of the third line, so these vectors are each scalar multiples of the third direction vector. Then the first two direction vectors are also scalar multiples of each other, so these vectors, and hence the two lines, are parallel.

(b) False; for example, the x- and y-axes are both perpendicular to the z-axis, yet the x- and y-axes are not parallel.

(c) True; each of the first two planes has a normal vector parallel to the normal vector of the third plane, so these two normal vectors are parallel to each other and the planes are parallel.

(d) False; for example, the xy- and yz-planes are not parallel, yet they are both perpendicular to the xz-plane.

(e) False; the x- and y-axes are not parallel, yet they are both parallel to the plane $z = 1$.

(f) True; if each line is perpendicular to a plane, then the lines' direction vectors are both parallel to a normal vector for the plane. Thus, the direction vectors are parallel to each other and the lines are parallel.

(g) False; the planes $y = 1$ and $z = 1$ are not parallel, yet they are both parallel to the x-axis.

(h) True; if each plane is perpendicular to a line, then any normal vector for each plane is parallel to a direction vector for the line. Thus, the normal vectors are parallel to each other and the planes are parallel.

(i) True; see Figure 9 and the accompanying discussion.

(j) False; they can be skew, as in Example 3.

(k) True. Consider any normal vector for the plane and any direction vector for the line. If the normal vector is perpendicular to the direction vector, the line and plane are parallel. Otherwise, the vectors meet at an angle θ, $0^\circ \le \theta < 90^\circ$, and the line will intersect the plane at an angle $90^\circ - \theta$.

2. For this line, we have $\mathbf{r}_0 = 6\,\mathbf{i} - 5\,\mathbf{j} + 2\,\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 3\,\mathbf{j} - \frac{2}{3}\,\mathbf{k}$, so a vector equation is $\mathbf{r} = \mathbf{r}_0 + t\,\mathbf{v} = (6\,\mathbf{i} - 5\,\mathbf{j} + 2\,\mathbf{k}) + t\left(\mathbf{i} + 3\,\mathbf{j} - \frac{2}{3}\,\mathbf{k}\right) = (6 + t)\,\mathbf{i} + (-5 + 3t)\,\mathbf{j} + \left(2 - \frac{2}{3}t\right)\mathbf{k}$ and parametric equations are $x = 6 + t$, $y = -5 + 3t$, $z = 2 - \frac{2}{3}t$.

3. For this line, we have $\mathbf{r}_0 = 2\,\mathbf{i} + 2.4\,\mathbf{j} + 3.5\,\mathbf{k}$ and $\mathbf{v} = 3\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$, so a vector equation is $\mathbf{r} = \mathbf{r}_0 + t\,\mathbf{v} = (2\,\mathbf{i} + 2.4\,\mathbf{j} + 3.5\,\mathbf{k}) + t(3\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}) = (2 + 3t)\,\mathbf{i} + (2.4 + 2t)\,\mathbf{j} + (3.5 - t)\,\mathbf{k}$ and parametric equations are $x = 2 + 3t$, $y = 2.4 + 2t$, $z = 3.5 - t$.

4. This line has the same direction as the given line, $\mathbf{v} = 2\,\mathbf{i} - 3\,\mathbf{j} + 9\,\mathbf{k}$. Here $\mathbf{r}_0 = 14\,\mathbf{j} - 10\,\mathbf{k}$, so a vector equation is $\mathbf{r} = (14\,\mathbf{j} - 10\,\mathbf{k}) + t(2\,\mathbf{i} - 3\,\mathbf{j} + 9\,\mathbf{k}) = 2t\,\mathbf{i} + (14 - 3t)\,\mathbf{j} + (-10 + 9t)\,\mathbf{k}$ and parametric equations are $x = 2t$, $y = 14 - 3t$, $z = -10 + 9t$.

5. A line perpendicular to the given plane has the same direction as a normal vector to the plane, such as $\mathbf{n} = \langle 1, 3, 1 \rangle$. So $\mathbf{r}_0 = \mathbf{i} + 6\,\mathbf{k}$, and we can take $\mathbf{v} = \mathbf{i} + 3\,\mathbf{j} + \mathbf{k}$. Then a vector equation is $\mathbf{r} = (\mathbf{i} + 6\,\mathbf{k}) + t(\mathbf{i} + 3\,\mathbf{j} + \mathbf{k}) = (1 + t)\,\mathbf{i} + 3t\,\mathbf{j} + (6 + t)\,\mathbf{k}$, and parametric equations are $x = 1 + t$, $y = 3t$, $z = 6 + t$.

6. The vector $\mathbf{v} = \langle 4-0, 3-0, -1-0 \rangle = \langle 4, 3, -1 \rangle$ is parallel to the line. Letting $P_0 = (0, 0, 0)$, parametric equations are $x = 0 + 4 \cdot t = 4t$, $y = 0 + 3 \cdot t = 3t$, $z = 0 + (-1) \cdot t = -t$, while symmetric equations are $\dfrac{x}{4} = \dfrac{y}{3} = \dfrac{z}{-1}$ or $\dfrac{x}{4} = \dfrac{y}{3} = -z$.

7. The vector $\mathbf{v} = \left\langle 2-0, 1-\frac{1}{2}, -3-1 \right\rangle = \left\langle 2, \frac{1}{2}, -4 \right\rangle$ is parallel to the line. Letting $P_0 = (2, 1, -3)$, parametric equations are $x = 2 + 2t$, $y = 1 + \frac{1}{2}t$, $z = -3 - 4t$, while symmetric equations are $\dfrac{x-2}{2} = \dfrac{y-1}{1/2} = \dfrac{z+3}{-4}$ or $\dfrac{x-2}{2} = 2y - 2 = \dfrac{z+3}{-4}$.

8. $\mathbf{v} = \langle 2.6-1.0, 1.2-2.4, 0.3-4.6 \rangle = \langle 1.6, -1.2, -4.3 \rangle$, and letting $P_0 = (1.0, 2.4, 4.6)$, parametric equations are $x = 1.0 + 1.6t$, $y = 2.4 - 1.2t$, $z = 4.6 - 4.3t$, while symmetric equations are $\dfrac{x-1.0}{1.6} = \dfrac{y-2.4}{-1.2} = \dfrac{z-4.6}{-4.3}$.

9. $\mathbf{v} = \langle 3-(-8), -2-1, 4-4 \rangle = \langle 11, -3, 0 \rangle$, and letting $P_0 = (-8, 1, 4)$, parametric equations are $x = -8 + 11t$, $y = 1 - 3t$, $z = 4 + 0t = 4$, while symmetric equations are $\dfrac{x+8}{11} = \dfrac{y-1}{-3}$, $z = 4$. Notice here that the direction number $c = 0$, so rather than writing $\dfrac{z-4}{0}$ in the symmetric equation we must write the equation $z = 4$ separately.

10. $\mathbf{v} = (\mathbf{i}+\mathbf{j}) \times (\mathbf{j}+\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ is the direction of the line perpendicular to both $\mathbf{i}+\mathbf{j}$ and $\mathbf{j}+\mathbf{k}$. With $P_0 = (2, 1, 0)$, parametric equations are $x = 2 + t$, $y = 1 - t$, $z = t$ and symmetric equations are $x - 2 = \dfrac{y-1}{-1} = z$ or $x - 2 = 1 - y = z$.

11. The line has direction $\mathbf{v} = \langle 1, 2, 1 \rangle$. Letting $P_0 = (1, -1, 1)$, parametric equations are $x = 1 + t$, $y = -1 + 2t$, $z = 1 + t$ and symmetric equations are $x - 1 = \dfrac{y+1}{2} = z - 1$.

12. Setting $z = 0$ we see that $(1, 0, 0)$ satisfies the equations of both planes, so they do in fact have a line of intersection. The line is perpendicular to the normal vectors of both planes, so a direction vector for the line is $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, 2, 3 \rangle \times \langle 1, -1, 1 \rangle = \langle 5, 2, -3 \rangle$. Taking the point $(1, 0, 0)$ as P_0, parametric equations are $x = 1 + 5t$, $y = 2t$, $z = -3t$, and symmetric equations are $\dfrac{x-1}{5} = \dfrac{y}{2} = \dfrac{z}{-3}$.

13. Direction vectors of the lines are $\mathbf{v}_1 = \langle -2-(-4), 0-(-6), -3-1 \rangle = \langle 2, 6, -4 \rangle$ and $\mathbf{v}_2 = \langle 5-10, 3-18, 14-4 \rangle = \langle -5, -15, 10 \rangle$, and since $\mathbf{v}_2 = -\frac{5}{2}\mathbf{v}_1$, the direction vectors and thus the lines are parallel.

14. Direction vectors of the lines are $\mathbf{v}_1 = \langle 3, -3, 1 \rangle$ and $\mathbf{v}_2 = \langle 1, -4, -12 \rangle$. Since $\mathbf{v}_1 \cdot \mathbf{v}_2 = 3 + 12 - 12 \neq 0$, the vectors and thus the lines are not perpendicular.

15. (a) The line passes through the point $(1, -5, 6)$ and a direction vector for the line is $\langle -1, 2, -3\rangle$, so symmetric equations for the line are $\dfrac{x-1}{-1} = \dfrac{y+5}{2} = \dfrac{z-6}{-3}$.

(b) The line intersects the xy-plane when $z = 0$, so we need $\dfrac{x-1}{-1} = \dfrac{y+5}{2} = \dfrac{0-6}{-3}$ or $\dfrac{x-1}{-1} = 2 \;\Rightarrow\; x = -1$, $\dfrac{y+5}{2} = 2 \;\Rightarrow\; y = -1$. Thus the point of intersection with the xy-plane is $(-1, -1, 0)$. Similarly for the yz-plane, we need $x = 0 \;\Rightarrow\; 1 = \dfrac{y+5}{2} = \dfrac{z-6}{-3} \;\Rightarrow\; y = -3$, $z = 3$. Thus the line intersects the yz-plane at $(0, -3, 3)$. For the xz-plane, we need $y = 0 \;\Rightarrow\; \dfrac{x-1}{-1} = \dfrac{5}{2} = \dfrac{z-6}{-3} \;\Rightarrow\; x = -\frac{3}{2}$, $z = -\frac{3}{2}$. So the line intersects the xz-plane at $\left(-\frac{3}{2}, 0, -\frac{3}{2}\right)$.

16. (a) A vector normal to the plane $x - y + 3z = 7$ is $\mathbf{n} = \langle 1, -1, 3\rangle$, and since the line is to be perpendicular to the plane, $\mathbf{n}$ is also a direction vector for the line. Thus parametric equations of the line are $x = 2 + t$, $y = 4 - t$, $z = 6 + 3t$.

(b) On the xy-plane, $z = 0$. So $z = 6 + 3t = 0 \;\Rightarrow\; t = -2$ in the parametric equations of the line, and therefore $x = 0$ and $y = 6$, giving the point of intersection $(0, 6, 0)$. For the yz-plane, $x = 0$ so we get the same point of interesection: $(0, 6, 0)$. For the xz-plane, $y = 0$ which implies $t = 4$, so $x = 6$ and $z = 18$ and the point of intersection is $(6, 0, 18)$.

17. From Equation 4, the line segment from $\mathbf{r}_0 = 2\,\mathbf{i} - \mathbf{j} + 4\,\mathbf{k}$ to $\mathbf{r}_1 = 4\,\mathbf{i} + 6\,\mathbf{j} + \mathbf{k}$ is $\mathbf{r}(t) = (1-t)\,\mathbf{r}_0 + t\,\mathbf{r}_1 = (1-t)(2\,\mathbf{i} - \mathbf{j} + 4\,\mathbf{k}) + t(4\,\mathbf{i} + 6\,\mathbf{j} + \mathbf{k}) = (2\,\mathbf{i} - \mathbf{j} + 4\,\mathbf{k}) + t(2\,\mathbf{i} + 7\,\mathbf{j} - 3\,\mathbf{k})$, $0 \le t \le 1$.

18. From Equation 4, the line segment from $\mathbf{r}_0 = 10\,\mathbf{i} + 3\,\mathbf{j} + \mathbf{k}$ to $\mathbf{r}_1 = 5\,\mathbf{i} + 6\,\mathbf{j} - 3\,\mathbf{k}$ is

$$\begin{aligned}\mathbf{r}(t) &= (1-t)\,\mathbf{r}_0 + t\,\mathbf{r}_1 = (1-t)(10\,\mathbf{i} + 3\,\mathbf{j} + \mathbf{k}) + t(5\,\mathbf{i} + 6\,\mathbf{j} - 3\,\mathbf{k})\\ &= (10\,\mathbf{i} + 3\,\mathbf{j} + \mathbf{k}) + t(-5\,\mathbf{i} + 3\,\mathbf{j} - 4\,\mathbf{k}), \quad 0 \le t \le 1.\end{aligned}$$

The corresponding parametric equations are $x = 10 - 5t$, $y = 3 + 3t$, $z = 1 - 4t$, $0 \le t \le 1$.

19. Since the direction vectors $\langle 2, -1, 3\rangle$ and $\langle 4, -2, 5\rangle$ are not scalar multiples of each other, the lines aren't parallel. For the lines to intersect, we must be able to find one value of t and one value of s that produce the same point from the respective parametric equations. Thus we need to satisfy the following three equations: $3 + 2t = 1 + 4s$, $4 - t = 3 - 2s$, $1 + 3t = 4 + 5s$. Solving the last two equations we get $t = 1$, $s = 0$ and checking, we see that these values don't satisfy the first equation. Thus the lines aren't parallel and don't intersect, so they must be skew lines.

20. Since the direction vectors are $\mathbf{v}_1 = \langle -12, 9, -3\rangle$ and $\mathbf{v}_2 = \langle 8, -6, 2\rangle$, we have $\mathbf{v}_1 = -\frac{3}{2}\mathbf{v}_2$ so the lines are parallel.

21. Since the direction vectors $\langle 1, -2, -3\rangle$ and $\langle 1, 3, -7\rangle$ aren't scalar multiples of each other, the lines aren't parallel. Parametric equations of the lines are L_1: $x = 2 + t$, $y = 3 - 2t$, $z = 1 - 3t$ and L_2: $x = 3 + s$, $y = -4 + 3s$, $z = 2 - 7s$. Thus, for the lines to intersect, the three equations $2 + t = 3 + s$, $3 - 2t = -4 + 3s$, and $1 - 3t = 2 - 7s$ must be satisfied simultaneously. Solving the first two equations gives $t = 2$, $s = 1$ and checking, we see that these values do satisfy the third equation, so the lines intersect when $t = 2$ and $s = 1$, that is, at the point $(4, -1, -5)$.

22. The direction vectors $\langle 1, -1, 3\rangle$ and $\langle 2, -2, 7\rangle$ are not parallel, so neither are the lines. Parametric equations for the lines are L_1: $x = t$, $y = 1 - t$, $z = 2 + 3t$ and L_2: $x = 2 + 2s$, $y = 3 - 2s$, $z = 7s$. Thus, for the lines to interesect, the three equations $t = 2 + 2s$, $1 - t = 3 - 2s$, and $2 + 3t = 7s$ must be satisfied simultaneously. Solving the last two equations gives $t = -10$, $s = -4$ and checking, we see that these values don't satisfy the first equation. Thus the lines aren't parallel and don't intersect, so they must be skew.

23. Since the plane is perpendicular to the vector $\langle 1, -2, 5\rangle$, we can take $\langle 1, -2, 5\rangle$ as a normal vector to the plane. $(0, 0, 0)$ is a point on the plane, so setting $a = 1$, $b = -2$, $c = 5$ and $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ in Equation 7 gives $1(x - 0) + (-2)(y - 0) + 5(z - 0) = 0$ or $x - 2y + 5z = 0$ as an equation of the plane.

24. $2\,\mathbf{i} + \mathbf{j} - \mathbf{k} = \langle 2, 1, -1\rangle$ is a normal vector to the plane and $(5, 3, 5)$ is a point on the plane, so setting $a = 2$, $b = 1$, $c = -1$, $x_0 = 5$, $y_0 = 3$, $z_0 = 5$ in Equation 7 gives $2(x - 5) + 1(y - 3) + (-1)(z - 5) = 0$ or $2x + y - z = 8$ as an equation of the plane.

25. $\mathbf{i} + 4\,\mathbf{j} + \mathbf{k} = \langle 1, 4, 1\rangle$ is a normal vector to the plane and $\left(-1, \frac{1}{2}, 3\right)$ is a point on the plane, so setting $a = 1$, $b = 4$, $c = 1$, $x_0 = -1$, $y_0 = \frac{1}{2}$, $z_0 = 3$ in Equation 7 gives $1[x - (-1)] + 4\left(y - \frac{1}{2}\right) + 1(z - 3) = 0$ or $x + 4y + z = 4$ as an equation of the plane.

26. Since the line is perpendicular to the plane, its direction vector $\langle 3, -1, 4\rangle$ is a normal vector to the plane. The point $(2, 0, 1)$ is on the plane, so an equation of the plane is $3(x - 2) + (-1)(y - 0) + 4(z - 1) = 0$ or $3x - y + 4z = 10$.

27. Since the two planes are parallel, they will have the same normal vectors. So we can take $\mathbf{n} = \langle 5, -1, -1\rangle$, and an equation of the plane is $5(x - 1) - 1[y - (-1)] - 1[z - (-1)] = 0$ or $5x - y - z = 7$.

28. Since the two planes are parallel, they will have the same normal vectors. A normal vector for the plane $z = x + y$ or $x + y - z = 0$ is $\mathbf{n} = \langle 1, 1, -1\rangle$, and an equation of the desired plane is $1(x - 2) + 1(y - 4) - 1(z - 6) = 0$ or $x + y - z = 0$ (the same plane!).

29. Since the two planes are parallel, they will have the same normal vectors. So we can take $\mathbf{n} = \langle 1, 1, 1\rangle$, and an equation of the plane is $1(x - 1) + 1\left(y - \frac{1}{2}\right) + 1\left(z - \frac{1}{3}\right) = 0$ or $x + y + z = \frac{11}{6}$ or $6x + 6y + 6z = 11$.

30. First, a normal vector for the plane $5x + 2y + z = 1$ is $\mathbf{n} = \langle 5, 2, 1\rangle$. A direction vector for the line is $\mathbf{v} = \langle 1, -1, -3\rangle$, and since $\mathbf{n} \cdot \mathbf{v} = 0$ we know the line is perpendicular to $\mathbf{n}$ and hence parallel to the plane. Thus, there is a parallel plane which contains the line. By putting $t = 0$, we know that the point $(1, 2, 4)$ is on the line and hence the new plane. We can use the same normal vector $\mathbf{n} = \langle 5, 2, 1\rangle$, so an equation of the plane is $5(x - 1) + 2(y - 2) + 1(z - 4) = 0$ or $5x + 2y + z = 13$.

31. Here the vectors $\mathbf{a} = \langle 1 - 0, 0 - 1, 1 - 1\rangle = \langle 1, -1, 0\rangle$ and $\mathbf{b} = \langle 1 - 0, 1 - 1, 0 - 1\rangle = \langle 1, 0, -1\rangle$ lie in the plane, so $\mathbf{a} \times \mathbf{b}$ is a normal vector to the plane. Thus, we can take $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle 1 - 0, 0 + 1, 0 + 1\rangle = \langle 1, 1, 1\rangle$. If P_0 is the point $(0, 1, 1)$, an equation of the plane is $1(x - 0) + 1(y - 1) + 1(z - 1) = 0$ or $x + y + z = 2$.

32. Here the vectors $\mathbf{a} = \langle 2, -4, 6 \rangle$ and $\mathbf{b} = \langle 5, 1, 3 \rangle$ lie in the plane, so $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle -12 - 6, 30 - 6, 2 + 20 \rangle = \langle -18, 24, 22 \rangle$ is a normal vector to the plane and an equation of the plane is $-18(x - 0) + 24(y - 0) + 22(z - 0) = 0$ or $-18x + 24y + 22z = 0$.

33. Here the vectors $\mathbf{a} = \langle 8 - 3, 2 - (-1), 4 - 2 \rangle = \langle 5, 3, 2 \rangle$ and $\mathbf{b} = \langle -1 - 3, -2 - (-1), -3 - 2 \rangle = \langle -4, -1, -5 \rangle$ lie in the plane, so a normal vector to the plane is $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle -15 + 2, -8 + 25, -5 + 12 \rangle = \langle -13, 17, 7 \rangle$ and an equation of the plane is $-13(x - 3) + 17[y - (-1)] + 7(z - 2) = 0$ or $-13x + 17y + 7z = -42$.

34. If we first find two nonparallel vectors in the plane, their cross product will be a normal vector to the plane. Since the given line lies in the plane, its direction vector $\mathbf{a} = \langle 3, 1, -1 \rangle$ is one vector in the plane. We can verify that the given point $(1, 2, 3)$ does not lie on this line, so to find another nonparallel vector $\mathbf{b}$ which lies in the plane, we can pick any point on the line and find a vector connecting the points. If we put $t = 0$, we see that $(0, 1, 2)$ is on the line, so $\mathbf{b} = \langle 1 - 0, 2 - 1, 3 - 2 \rangle = \langle 1, 1, 1 \rangle$ and $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle 1 + 1, -1 - 3, 3 - 1 \rangle = \langle 2, -4, 2 \rangle$. Thus, an equation of the plane is $2(x - 1) - 4(y - 2) + 2(z - 3) = 0$ or $2x - 4y + 2z = 0$. (Equivalently, we can write $x - 2y + z = 0$.)

35. If we first find two nonparallel vectors in the plane, their cross product will be a normal vector to the plane. Since the given line lies in the plane, its direction vector $\mathbf{a} = \langle -2, 5, 4 \rangle$ is one vector in the plane. We can verify that the given point $(6, 0, -2)$ does not lie on this line, so to find another nonparallel vector $\mathbf{b}$ which lies in the plane, we can pick any point on the line and find a vector connecting the points. If we put $t = 0$, we see that $(4, 3, 7)$ is on the line, so $\mathbf{b} = \langle 6 - 4, 0 - 3, -2 - 7 \rangle = \langle 2, -3, -9 \rangle$ and $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle -45 + 12, 8 - 18, 6 - 10 \rangle = \langle -33, -10, -4 \rangle$. Thus, an equation of the plane is $-33(x - 6) - 10(y - 0) - 4[z - (-2)] = 0$ or $33x + 10y + 4z = 190$.

36. Since the line $x = 2y = 3z$, or $x = \dfrac{y}{1/2} = \dfrac{z}{1/3}$, lies in the plane, its direction vector $\mathbf{a} = \left\langle 1, \frac{1}{2}, \frac{1}{3} \right\rangle$ is parallel to the plane. The point $(0, 0, 0)$ is on the line (put $t = 0$), and we can verify that the given point $(1, -1, 1)$ in the plane is not on the line. The vector connecting these two points, $\mathbf{b} = \langle 1, -1, 1 \rangle$, is therefore parallel to the plane, but not parallel to $\langle 1, 2, 3 \rangle$. Then $\mathbf{a} \times \mathbf{b} = \left\langle \frac{1}{2} + \frac{1}{3}, \frac{1}{3} - 1, -1 - \frac{1}{2} \right\rangle = \left\langle \frac{5}{6}, -\frac{2}{3}, -\frac{3}{2} \right\rangle$ is a normal vector to the plane, and an equation of the plane is $\frac{5}{6}(x - 0) - \frac{2}{3}(y - 0) - \frac{3}{2}(z - 0) = 0$ or $5x - 4y - 9z = 0$.

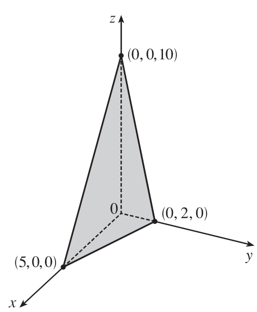
37. A direction vector for the line of intersection is $\mathbf{a} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, 1, -1 \rangle \times \langle 2, -1, 3 \rangle = \langle 2, -5, -3 \rangle$, and $\mathbf{a}$ is parallel to the desired plane. Another vector parallel to the plane is the vector connecting any point on the line of intersection to the given point $(-1, 2, 1)$ in the plane. Setting $x = 0$, the equations of the planes reduce to $y - z = 2$ and $-y + 3z = 1$ with simultaneous solution $y = \frac{7}{2}$ and $z = \frac{3}{2}$. So a point on the line is $\left(0, \frac{7}{2}, \frac{3}{2}\right)$ and another vector parallel to the plane is $\left\langle -1, -\frac{3}{2}, -\frac{1}{2} \right\rangle$. Then a normal vector to the plane is $\mathbf{n} = \langle 2, -5, -3 \rangle \times \left\langle -1, -\frac{3}{2}, -\frac{1}{2} \right\rangle = \langle -2, 4, -8 \rangle$ and an equation of the plane is $-2(x + 1) + 4(y - 2) - 8(z - 1) = 0$ or $x - 2y + 4z = -1$.

38. The points $(0, -2, 5)$ and $(-1, 3, 1)$ lie in the desired plane, so the vector $\mathbf{v}_1 = \langle -1, 5, -4\rangle$ connecting them is parallel to the plane. The desired plane is perpendicular to the plane $2z = 5x + 4y$ or $5x + 4y - 2z = 0$ and for perpendicular planes, a normal vector for one plane is parallel to the other plane, so $\mathbf{v}_2 = \langle 5, 4, -2\rangle$ is also parallel to the desired plane. A normal vector to the desired plane is $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle -10 + 16, -20 - 2, -4 - 25\rangle = \langle 6, -22, -29\rangle$. Taking $(x_0, y_0, z_0) = (0, -2, 5)$, the equation we are looking for is $6(x - 0) - 22(y + 2) - 29(z - 5) = 0$ or $6x - 22y - 29z = -101$.

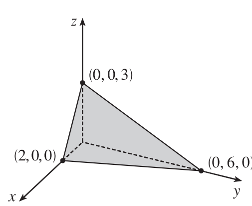
39. If a plane is perpendicular to two other planes, its normal vector is perpendicular to the normal vectors of the other two planes. Thus $\langle 2, 1, -2\rangle \times \langle 1, 0, 3\rangle = \langle 3 - 0, -2 - 6, 0 - 1\rangle = \langle 3, -8, -1\rangle$ is a normal vector to the desired plane. The point $(1, 5, 1)$ lies on the plane, so an equation is $3(x - 1) - 8(y - 5) - (z - 1) = 0$ or $3x - 8y - z = -38$.

40. $\mathbf{n}_1 = \langle 1, 0, -1\rangle$ and $\mathbf{n}_2 = \langle 0, 1, 2\rangle$. Setting $z = 0$, it is easy to see that $(1, 3, 0)$ is a point on the line of intersection of $x - z = 1$ and $y + 2z = 3$. The direction of this line is $\mathbf{v}_1 = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, -2, 1\rangle$. A second vector parallel to the desired plane is $\mathbf{v}_2 = \langle 1, 1, -2\rangle$, since it is perpendicular to $x + y - 2z = 1$. Therefore, a normal of the plane in question is $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 4 - 1, 1 + 2, 1 + 2\rangle = \langle 3, 3, 3\rangle$, or we can use $\langle 1, 1, 1\rangle$. Taking $(x_0, y_0, z_0) = (1, 3, 0)$, the equation we are looking for is $(x - 1) + (y - 3) + z = 0 \quad\Leftrightarrow\quad x + y + z = 4$.

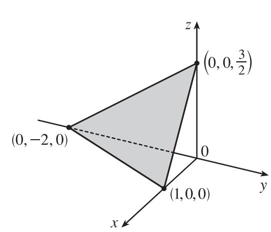
41. To find the x-intercept we set $y = z = 0$ in the equation $2x + 5y + z = 10$ and obtain $2x = 10 \;\Rightarrow\; x = 5$ so the x-intercept is $(5, 0, 0)$. When $x = z = 0$ we get $5y = 10 \;\Rightarrow\; y = 2$, so the y-intercept is $(0, 2, 0)$. Setting $x = y = 0$ gives $z = 10$, so the z-intercept is $(0, 0, 10)$ and we graph the portion of the plane that lies in the first octant.



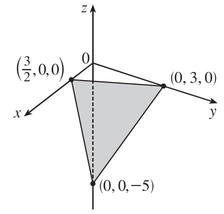
42. To find the x-intercept we set $y = z = 0$ in the equation $3x + y + 2z = 6$ and obtain $3x = 6 \;\Rightarrow\; x = 2$ so the x-intercept is $(2, 0, 0)$. When $x = z = 0$ we get $y = 6$ so the y-intercept is $(0, 6, 0)$. Setting $x = y = 0$ gives $2z = 6 \;\Rightarrow\; z = 3$, so the z-intercept is $(0, 0, 3)$. The figure shows the portion of the plane that lies in the first octant.



43. Setting $y = z = 0$ in the equation $6x - 3y + 4z = 6$ gives $6x = 6 \;\Rightarrow\; x = 1$, when $x = z = 0$ we have $-3y = 6 \;\Rightarrow\; y = -2$, and $x = y = 0$ implies $4z = 6 \;\Rightarrow\; z = \frac{3}{2}$, so the intercepts are $(1, 0, 0)$, $(0, -2, 0)$, and $(0, 0, \frac{3}{2})$. The figure shows the portion of the plane cut off by the coordinate planes.



44. Setting $y = z = 0$ in the equation $6x + 5y - 3z = 15$ gives $6x = 15 \Rightarrow$ $x = \frac{5}{2}$, when $x = z = 0$ we have $5y = 15 \Rightarrow y = 3$, and $x = y = 0$ implies $-3z = 15 \Rightarrow z = -5$, so the intercepts are $(\frac{5}{2}, 0, 0)$, $(0, 3, 0)$, and $(0, 0, -5)$. The figure shows the portion of the plane cut off by the coordinate planes.



45. Substitute the parametric equations of the line into the equation of the plane: $(3 - t) - (2 + t) + 2(5t) = 9 \Rightarrow$ $8t = 8 \Rightarrow t = 1$. Therefore, the point of intersection of the line and the plane is given by $x = 3 - 1 = 2$, $y = 2 + 1 = 3$, and $z = 5(1) = 5$, that is, the point $(2, 3, 5)$.

46. Substitute the parametric equations of the line into the equation of the plane: $(1 + 2t) + 2(4t) - (2 - 3t) + 1 = 0 \Rightarrow$ $13t = 0 \Rightarrow t = 0$. Therefore, the point of intersection of the line and the plane is given by $x = 1 + 2(0) = 1$, $y = 4(0) = 0$, and $z = 2 - 3(0) = 2$, that is, the point $(1, 0, 2)$.

47. Parametric equations for the line are $x = t$, $y = 1 + t$, $z = \frac{1}{2}t$ and substituting into the equation of the plane gives $4(t) - (1 + t) + 3(\frac{1}{2}t) = 8 \Rightarrow \frac{9}{2}t = 9 \Rightarrow t = 2$. Thus $x = 2$, $y = 1 + 2 = 3$, $z = \frac{1}{2}(2) = 1$ and the point of intersection is $(2, 3, 1)$.

48. A direction vector for the line through $(1, 0, 1)$ and $(4, -2, 2)$ is $\mathbf{v} = \langle 3, -2, 1 \rangle$ and, taking $P_0 = (1, 0, 1)$, parametric equations for the line are $x = 1 + 3t$, $y = -2t$, $z = 1 + t$. Substitution of the parametric equations into the equation of the plane gives $1 + 3t - 2t + 1 + t = 6 \Rightarrow t = 2$. Then $x = 1 + 3(2) = 7$, $y = -2(2) = -4$, and $z = 1 + 2 = 3$ so the point of intersection is $(7, -4, 3)$.

49. Setting $x = 0$, we see that $(0, 1, 0)$ satisfies the equations of both planes, so that they do in fact have a line of intersection. $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, 1, 1 \rangle \times \langle 1, 0, 1 \rangle = \langle 1, 0, -1 \rangle$ is the direction of this line. Therefore, direction numbers of the intersecting line are $1, 0, -1$.

50. The angle between the two planes is the same as the angle between their normal vectors. The normal vectors of the two planes are $\langle 1, 1, 1 \rangle$ and $\langle 1, 2, 3 \rangle$. The cosine of the angle θ between these two planes is

$$\cos\theta = \frac{\langle 1,1,1 \rangle \cdot \langle 1,2,3 \rangle}{|\langle 1,1,1 \rangle|\,|\langle 1,2,3 \rangle|} = \frac{1+2+3}{\sqrt{1+1+1}\,\sqrt{1+4+9}} = \frac{6}{\sqrt{42}} = \sqrt{\frac{6}{7}}.$$

51. Normal vectors for the planes are $\mathbf{n}_1 = \langle 1, 4, -3 \rangle$ and $\mathbf{n}_2 = \langle -3, 6, 7 \rangle$, so the normals (and thus the planes) aren't parallel. But $\mathbf{n}_1 \cdot \mathbf{n}_2 = -3 + 24 - 21 = 0$, so the normals (and thus the planes) are perpendicular.

52. Normal vectors for the planes are $\mathbf{n}_1 = \langle -1, 4, -2 \rangle$ and $\mathbf{n}_2 = \langle 3, -12, 6 \rangle$. Since $\mathbf{n}_2 = -3\mathbf{n}_1$, the normals (and thus the planes) are parallel.

53. Normal vectors for the planes are $\mathbf{n}_1 = \langle 1, 1, 1 \rangle$ and $\mathbf{n}_2 = \langle 1, -1, 1 \rangle$. The normals are not parallel, so neither are the planes. Furthermore, $\mathbf{n}_1 \cdot \mathbf{n}_2 = 1 - 1 + 1 = 1 \neq 0$, so the planes aren't perpendicular. The angle between them is given by

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \frac{1}{\sqrt{3}\,\sqrt{3}} = \frac{1}{3} \quad\Rightarrow\quad \theta = \cos^{-1}\!\left(\tfrac{1}{3}\right) \approx 70.5^\circ.$$

54. The normals are $\mathbf{n}_1 = \langle 2, -3, 4\rangle$ and $\mathbf{n}_2 = \langle 1, 6, 4\rangle$ so the planes aren't parallel. Since $\mathbf{n}_1 \cdot \mathbf{n}_2 = 2 - 18 + 16 = 0$, the normals (and thus the planes) are perpendicular.

55. The normals are $\mathbf{n}_1 = \langle 1, -4, 2\rangle$ and $\mathbf{n}_2 = \langle 2, -8, 4\rangle$. Since $\mathbf{n}_2 = 2\mathbf{n}_1$, the normals (and thus the planes) are parallel.

56. The normal vectors are $\mathbf{n}_1 = \langle 1, 2, 2\rangle$ and $\mathbf{n}_2 = \langle 2, -1, 2\rangle$. The normals are not parallel, so neither are the planes. Furthermore, $\mathbf{n}_1 \cdot \mathbf{n}_2 = 2 - 2 + 4 = 4 \neq 0$, so the planes aren't perpendicular. The angle between them is given by

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \frac{4}{\sqrt{9}\,\sqrt{9}} = \frac{4}{9} \quad\Rightarrow\quad \theta = \cos^{-1}\left(\tfrac{4}{9}\right) \approx 63.6^\circ.$$

57. (a) To find a point on the line of intersection, set one of the variables equal to a constant, say $z = 0$. (This will fail if the line of intersection does not cross the xy-plane; in that case, try setting x or y equal to 0.) The equations of the two planes reduce to $x + y = 1$ and $x + 2y = 1$. Solving these two equations gives $x = 1$, $y = 0$. Thus a point on the line is $(1, 0, 0)$. A vector $\mathbf{v}$ in the direction of this intersecting line is perpendicular to the normal vectors of both planes, so we can take $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, 1, 1\rangle \times \langle 1, 2, 2\rangle = \langle 2 - 2, 1 - 2, 2 - 1\rangle = \langle 0, -1, 1\rangle$. By Equations 2, parametric equations for the line are $x = 1$, $y = -t$, $z = t$.

(b) The angle between the planes satisfies $\cos\theta = \dfrac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \dfrac{1 + 2 + 2}{\sqrt{3}\,\sqrt{9}} = \dfrac{5}{3\sqrt{3}}$. Therefore $\theta = \cos^{-1}\left(\dfrac{5}{3\sqrt{3}}\right) \approx 15.8^\circ$.

58. (a) If we set $z = 0$ then the equations of the planes reduce to $3x - 2y = 1$ and $2x + y = 3$ and solving these two equations gives $x = 1$, $y = 1$. Thus a point on the line of intersection is $(1, 1, 0)$. A vector $\mathbf{v}$ in the direction of this intersecting line is perpendicular to the normal vectors of both planes, so let $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 3, -2, 1\rangle \times \langle 2, 1, -3\rangle = \langle 5, 11, 7\rangle$. By Equations 2, parametric equations for the line are $x = 1 + 5t$, $y = 1 + 11t$, $z = 7t$.

(b) $\cos\theta = \dfrac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \dfrac{6 - 2 - 3}{\sqrt{14}\,\sqrt{14}} = \dfrac{1}{14} \quad\Rightarrow\quad \theta = \cos^{-1}\left(\tfrac{1}{14}\right) \approx 85.9^\circ$.

59. Setting $z = 0$, the equations of the two planes become $5x - 2y = 1$ and $4x + y = 6$. Solving these two equations gives $x = 1$, $y = 2$ so a point on the line of intersection is $(1, 2, 0)$. A vector $\mathbf{v}$ in the direction of this intersecting line is perpendicular to the normal vectors of both planes. So we can use $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 5, -2, -2\rangle \times \langle 4, 1, 1\rangle = \langle 0, -13, 13\rangle$ or equivalently we can take $\mathbf{v} = \langle 0, -1, 1\rangle$, and symmetric equations for the line are $x = 1$, $\dfrac{y - 2}{-1} = \dfrac{z}{1}$ or $x = 1$, $y - 2 = -z$.

60. If we set $z = 0$ then the equations of the planes reduce to $2x - y - 5 = 0$ and $4x + 3y - 5 = 0$ and solving these two equations gives $x = 2$, $y = -1$. Thus a point on the line of intersection is $(2, -1, 0)$. A vector $\mathbf{v}$ in the direction of this intersecting line is perpendicular to the normal vectors of both planes, so take $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 2, -1, -1\rangle \times \langle 4, 3, -1\rangle = \langle 4, -2, 10\rangle$ or equivalently we can take $\mathbf{v} = \langle 2, -1, 5\rangle$. Symmetric equations for the line are $\dfrac{x - 2}{2} = \dfrac{y + 1}{-1} = \dfrac{z}{5}$.

61. The distance from a point (x, y, z) to $(1, 0, -2)$ is $d_1 = \sqrt{(x - 1)^2 + y^2 + (z + 2)^2}$ and the distance from (x, y, z) to $(3, 4, 0)$ is $d_2 = \sqrt{(x - 3)^2 + (y - 4)^2 + z^2}$. The plane consists of all points (x, y, z) where $d_1 = d_2 \quad\Rightarrow\quad d_1^2 = d_2^2 \quad\Leftrightarrow$

$(x-1)^2 + y^2 + (z+2)^2 = (x-3)^2 + (y-4)^2 + z^2 \quad \Leftrightarrow$

$x^2 - 2x + y^2 + z^2 + 4z + 5 = x^2 - 6x + y^2 - 8y + z^2 + 25 \quad \Leftrightarrow \quad 4x + 8y + 4z = 20$ so an equation for the plane is $4x + 8y + 4z = 20$ or equivalently $x + 2y + z = 5$.

Alternatively, you can argue that the segment joining points $(1, 0, -2)$ and $(3, 4, 0)$ is perpendicular to the plane and the plane includes the midpoint of the segment.

62. The distance from a point (x, y, z) to $(2, 5, 5)$ is $d_1 = \sqrt{(x-2)^2 + (y-5)^2 + (z-5)^2}$ and the distance from (x, y, z) to $(-6, 3, 1)$ is $d_2 = \sqrt{(x+6)^2 + (y-3)^2 + (z-1)^2}$. The plane consists of all points (x, y, z) where $d_1 = d_2 \quad \Rightarrow$

$d_1^2 = d_2^2 \quad \Leftrightarrow \quad (x-2)^2 + (y-5)^2 + (z-5)^2 = (x+6)^2 + (y-3)^2 + (z-1)^2 \quad \Leftrightarrow$

$x^2 - 4x + y^2 - 10y + z^2 - 10z + 54 = x^2 + 12x + y^2 - 6y + z^2 - 2z + 46 \quad \Leftrightarrow \quad 16x + 4y + 8z = 8$ so an equation for the plane is $16x + 4y + 8z = 8$ or equivalently $4x + y + 2z = 2$.

63. The plane contains the points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$. Thus the vectors $\mathbf{a} = \langle -a, b, 0 \rangle$ and $\mathbf{b} = \langle -a, 0, c \rangle$ lie in the plane, and $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle bc - 0, 0 + ac, 0 + ab \rangle = \langle bc, ac, ab \rangle$ is a normal vector to the plane. The equation of the plane is therefore $bcx + acy + abz = abc + 0 + 0$ or $bcx + acy + abz = abc$. Notice that if $a \neq 0$, $b \neq 0$ and $c \neq 0$ then we can rewrite the equation as $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$. This is a good equation to remember!

64. (a) For the lines to intersect, we must be able to find one value of t and one value of s satisfying the three equations $1 + t = 2 - s$, $1 - t = s$ and $2t = 2$. From the third we get $t = 1$, and putting this in the second gives $s = 0$. These values of s and t do satisfy the first equation, so the lines intersect at the point $P_0 = (1 + 1, 1 - 1, 2(1)) = (2, 0, 2)$.

(b) The direction vectors of the lines are $\langle 1, -1, 2 \rangle$ and $\langle -1, 1, 0 \rangle$, so a normal vector for the plane is $\langle -1, 1, 0 \rangle \times \langle 1, -1, 2 \rangle = \langle 2, 2, 0 \rangle$ and it contains the point $(2, 0, 2)$. Then an equation of the plane is $2(x-2) + 2(y-0) + 0(z-2) = 0 \quad \Leftrightarrow \quad x + y = 2$.

65. Two vectors which are perpendicular to the required line are the normal of the given plane, $\langle 1, 1, 1 \rangle$, and a direction vector for the given line, $\langle 1, -1, 2 \rangle$. So a direction vector for the required line is $\langle 1, 1, 1 \rangle \times \langle 1, -1, 2 \rangle = \langle 3, -1, -2 \rangle$. Thus L is given by $\langle x, y, z \rangle = \langle 0, 1, 2 \rangle + t\langle 3, -1, -2 \rangle$, or in parametric form, $x = 3t$, $y = 1 - t$, $z = 2 - 2t$.

66. Let L be the given line. Then $(1, 1, 0)$ is the point on L corresponding to $t = 0$. L is in the direction of $\mathbf{a} = \langle 1, -1, 2 \rangle$ and $\mathbf{b} = \langle -1, 0, 2 \rangle$ is the vector joining $(1, 1, 0)$ and $(0, 1, 2)$. Then

$\mathbf{b} - \operatorname{proj}_{\mathbf{a}} \mathbf{b} = \langle -1, 0, 2 \rangle - \dfrac{\langle 1, -1, 2 \rangle \cdot \langle -1, 0, 2 \rangle}{1^2 + (-1)^2 + 2^2} \langle 1, -1, 2 \rangle = \langle -1, 0, 2 \rangle - \frac{1}{2}\langle 1, -1, 2 \rangle = \left\langle -\frac{3}{2}, \frac{1}{2}, 1 \right\rangle$ is a direction vector for the required line. Thus $2\left\langle -\frac{3}{2}, \frac{1}{2}, 1 \right\rangle = \langle -3, 1, 2 \rangle$ is also a direction vector, and the line has parametric equations $x = -3t$, $y = 1 + t$, $z = 2 + 2t$. (Notice that this is the same line as in Exercise 65.)

67. Let P_i have normal vector $\mathbf{n}_i$. Then $\mathbf{n}_1 = \langle 3, 6, -3 \rangle$, $\mathbf{n}_2 = \langle 4, -12, 8 \rangle$, $\mathbf{n}_3 = \langle 3, -9, 6 \rangle$, $\mathbf{n}_4 = \langle 1, 2, -1 \rangle$. Now $\mathbf{n}_1 = 3\mathbf{n}_4$, so $\mathbf{n}_1$ and $\mathbf{n}_4$ are parallel, and hence P_1 and P_4 are parallel; similarly P_2 and P_3 are parallel because $\mathbf{n}_2 = \frac{4}{3}\mathbf{n}_3$. However, $\mathbf{n}_1$ and $\mathbf{n}_2$ are not parallel (so not all four planes are parallel). Notice that the point $(2, 0, 0)$ lies on both P_1 and P_4, so these two planes are identical. The point $\left(\frac{5}{4}, 0, 0\right)$ lies on P_2 but not on P_3, so these are different planes.

68. Let L_i have direction vector $\mathbf{v}_i$. Rewrite the symmetric equations for L_3 as $\dfrac{x-1}{1/2} = \dfrac{y-1}{-1/4} = \dfrac{z+1}{1}$; then $\mathbf{v}_1 = \langle 6, -3, 12 \rangle$, $\mathbf{v}_2 = \langle 2, 1, 4 \rangle$, $\mathbf{v}_3 = \left\langle \frac{1}{2}, -\frac{1}{4}, 1 \right\rangle$, and $\mathbf{v}_4 = \langle 4, 2, 8 \rangle$. $\mathbf{v}_1 = 12\mathbf{v}_3$, so L_1 and L_3 are parallel. $\mathbf{v}_4 = 2\mathbf{v}_2$, so L_2 and L_4 are parallel. (Note that L_1 and L_2 are not parallel.) L_1 contains the point $(1, 1, 5)$, but this point does not lie on L_3, so they're not identical. $(3, 1, 5)$ lies on L_4 and also on L_2 (for $t = 1$), so L_2 and L_4 are the same line.

69. Let $Q = (1, 3, 4)$ and $R = (2, 1, 1)$, points on the line corresponding to $t = 0$ and $t = 1$. Let $P = (4, 1, -2)$. Then $\mathbf{a} = \overrightarrow{QR} = \langle 1, -2, -3 \rangle$, $\mathbf{b} = \overrightarrow{QP} = \langle 3, -2, -6 \rangle$. The distance is

$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|} = \frac{|\langle 1, -2, -3 \rangle \times \langle 3, -2, -6 \rangle|}{|\langle 1, -2, -3 \rangle|} = \frac{|\langle 6, -3, 4 \rangle|}{|\langle 1, -2, -3 \rangle|} = \frac{\sqrt{6^2 + (-3)^2 + 4^2}}{\sqrt{1^2 + (-2)^2 + (-3)^2}} = \frac{\sqrt{61}}{\sqrt{14}} = \sqrt{\frac{61}{14}}.$$

70. Let $Q = (0, 6, 3)$ and $R = (2, 4, 4)$, points on the line corresponding to $t = 0$ and $t = 1$. Let $P = (0, 1, 3)$. Then $\mathbf{a} = \overrightarrow{QR} = \langle 2, -2, 1 \rangle$ and $\mathbf{b} = \overrightarrow{QP} = \langle 0, -5, 0 \rangle$. The distance is

$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|} = \frac{|\langle 2, -2, 1 \rangle \times \langle 0, -5, 0 \rangle|}{|\langle 2, -2, 1 \rangle|} = \frac{|\langle 5, 0, -10 \rangle|}{|\langle 2, -2, 1 \rangle|} = \frac{\sqrt{5^2 + 0^2 + (-10)^2}}{\sqrt{2^2 + (-2)^2 + 1^2}} = \frac{\sqrt{125}}{\sqrt{9}} = \frac{5\sqrt{5}}{3}.$$

71. By Equation 9, the distance is $D = \dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} = \dfrac{|3(1) + 2(-2) + 6(4) - 5|}{\sqrt{3^2 + 2^2 + 6^2}} = \dfrac{|18|}{\sqrt{49}} = \dfrac{18}{7}$.

72. By Equation 9, the distance is $D = \dfrac{|1(-6) - 2(3) - 4(5) - 8|}{\sqrt{1^2 + (-2)^2 + (-4)^2}} = \dfrac{|-40|}{\sqrt{21}} = \dfrac{40}{\sqrt{21}}$.

73. Put $y = z = 0$ in the equation of the first plane to get the point $(2, 0, 0)$ on the plane. Because the planes are parallel, the distance D between them is the distance from $(2, 0, 0)$ to the second plane. By Equation 9,

$$D = \frac{|4(2) - 6(0) + 2(0) - 3|}{\sqrt{4^2 + (-6)^2 + (2)^2}} = \frac{5}{\sqrt{56}} = \frac{5}{2\sqrt{14}} \text{ or } \frac{5\sqrt{14}}{28}.$$

74. Put $x = y = 0$ in the equation of the first plane to get the point $(0, 0, 0)$ on the plane. Because the planes are parallel the distance D between them is the distance from $(0, 0, 0)$ to the second plane $3x - 6y + 9z - 1 = 0$. By Equation 9,

$$D = \frac{|3(0) - 6(0) + 9(0) - 1|}{\sqrt{3^2 + (-6)^2 + 9^2}} = \frac{1}{\sqrt{126}} = \frac{1}{3\sqrt{14}}.$$

75. The distance between two parallel planes is the same as the distance between a point on one of the planes and the other plane. Let $P_0 = (x_0, y_0, z_0)$ be a point on the plane given by $ax + by + cz + d_1 = 0$. Then $ax_0 + by_0 + cz_0 + d_1 = 0$ and the distance between P_0 and the plane given by $ax + by + cz + d_2 = 0$ is, from Equation 9,

$$D = \frac{|ax_0 + by_0 + cz_0 + d_2|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|-d_1 + d_2|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}.$$

76. The planes must have parallel normal vectors, so if $ax + by + cz + d = 0$ is such a plane, then for some $t \neq 0$, $\langle a, b, c \rangle = t\langle 1, 2, -2 \rangle = \langle t, 2t, -2t \rangle$. So this plane is given by the equation $x + 2y - 2z + k = 0$, where $k = d/t$. By Exercise 75, the distance between the planes is $2 = \dfrac{|1 - k|}{\sqrt{1^2 + 2^2 + (-2)^2}} \Leftrightarrow 6 = |1 - k| \Leftrightarrow k = 7$ or -5. So the desired planes have equations $x + 2y - 2z = 7$ and $x + 2y - 2z = -5$.

77. L_1: $x = y = z \Rightarrow x = y$ **(1)**. L_2: $x + 1 = y/2 = z/3 \Rightarrow x + 1 = y/2$ **(2)**. The solution of **(1)** and **(2)** is $x = y = -2$. However, when $x = -2$, $x = z \Rightarrow z = -2$, but $x + 1 = z/3 \Rightarrow z = -3$, a contradiction. Hence the lines do not intersect. For L_1, $\mathbf{v}_1 = \langle 1, 1, 1 \rangle$, and for L_2, $\mathbf{v}_2 = \langle 1, 2, 3 \rangle$, so the lines are not parallel. Thus the lines are skew lines. If two lines are skew, they can be viewed as lying in two parallel planes and so the distance between the skew lines would be the same as the distance between these parallel planes. The common normal vector to the planes must be perpendicular to both $\langle 1, 1, 1 \rangle$ and $\langle 1, 2, 3 \rangle$, the direction vectors of the two lines. So set $\mathbf{n} = \langle 1, 1, 1 \rangle \times \langle 1, 2, 3 \rangle = \langle 3 - 2, -3 + 1, 2 - 1 \rangle = \langle 1, -2, 1 \rangle$. From above, we know that $(-2, -2, -2)$ and $(-2, -2, -3)$ are points of L_1 and L_2 respectively. So in the notation of Equation 8, $1(-2) - 2(-2) + 1(-2) + d_1 = 0 \Rightarrow d_1 = 0$ and $1(-2) - 2(-2) + 1(-3) + d_2 = 0 \Rightarrow d_2 = 1$.

By Exercise 75, the distance between these two skew lines is $D = \dfrac{|0 - 1|}{\sqrt{1 + 4 + 1}} = \dfrac{1}{\sqrt{6}}$.

Alternate solution (without reference to planes): A vector which is perpendicular to both of the lines is $\mathbf{n} = \langle 1, 1, 1 \rangle \times \langle 1, 2, 3 \rangle = \langle 1, -2, 1 \rangle$. Pick any point on each of the lines, say $(-2, -2, -2)$ and $(-2, -2, -3)$, and form the vector $\mathbf{b} = \langle 0, 0, 1 \rangle$ connecting the two points. The distance between the two skew lines is the absolute value of the scalar projection of $\mathbf{b}$ along $\mathbf{n}$, that is, $D = \dfrac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} = \dfrac{|1 \cdot 0 - 2 \cdot 0 + 1 \cdot 1|}{\sqrt{1 + 4 + 1}} = \dfrac{1}{\sqrt{6}}$.

78. First notice that if two lines are skew, they can be viewed as lying in two parallel planes and so the distance between the skew lines would be the same as the distance between these parallel planes. The common normal vector to the planes must be perpendicular to both $\mathbf{v}_1 = \langle 1, 6, 2 \rangle$ and $\mathbf{v}_2 = \langle 2, 15, 6 \rangle$, the direction vectors of the two lines respectively. Thus set $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 36 - 30, 4 - 6, 15 - 12 \rangle = \langle 6, -2, 3 \rangle$. Setting $t = 0$ and $s = 0$ gives the points $(1, 1, 0)$ and $(1, 5, -2)$. So in the notation of Equation 8, $6 - 2 + 0 + d_1 = 0 \Rightarrow d_1 = -4$ and $6 - 10 - 6 + d_2 = 0 \Rightarrow d_2 = 10$.

Then by Exercise 75, the distance between the two skew lines is given by $D = \dfrac{|-4 - 10|}{\sqrt{36 + 4 + 9}} = \dfrac{14}{7} = 2$.

Alternate solution (without reference to planes): We already know that the direction vectors of the two lines are $\mathbf{v}_1 = \langle 1, 6, 2 \rangle$ and $\mathbf{v}_2 = \langle 2, 15, 6 \rangle$. Then $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 6, -2, 3 \rangle$ is perpendicular to both lines. Pick any point on

0.0.3 Questions Solutions on Absolute Max and Min

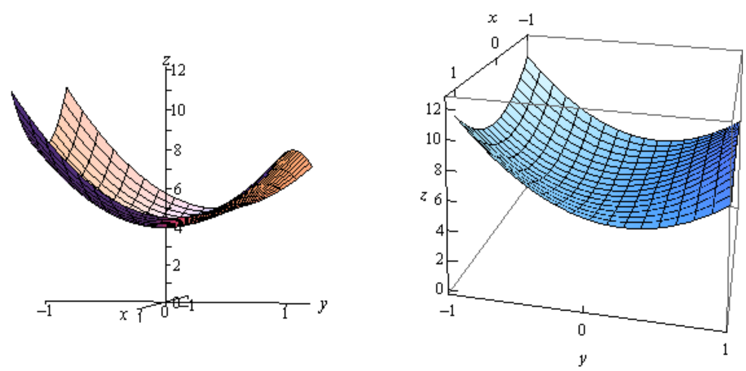
$$f(0,0) = 4 \qquad f(1,-1) = 11 \qquad f(1,1) = 7$$

$$f\left(1, \frac{1}{4}\right) = 4.75 \qquad f(-1,1) = 7 \qquad f(-1,-1) = 11$$

$$f\left(-1, \frac{1}{4}\right) = 4.75 \qquad f(0,1) = 8 \qquad f(0,-1) = 8$$

The absolute minimum is at $(0,0)$ since gives the smallest function value and the absolute maximum occurs at $(1,-1)$ and $(-1,-1)$ since these two points give the largest function value.

Here is a sketch of the function on the rectangle for reference purposes.



As this example has shown these can be very long problems on occasion. Let's take a look at an easier, well shorter anyway, problem with a different kind of boundary.

Example 2 Find the absolute minimum and absolute maximum of $f(x,y) = 2x^2 - y^2 + 6y$ on the disk of radius 4, $x^2 + y^2 \leq 16$

First note that a disk of radius 4 is given by the inequality in the problem statement. The "less than" inequality is included to get the interior of the disk and the equal sign is included to get the boundary. Of course, this also means that the boundary of the disk is a circle of radius 4.

Let's first find the critical points of the function that lies inside the disk. This will require the following two first order partial derivatives.

$$f_x = 4x \qquad f_y = -2y + 6$$

To find the critical points we'll need to solve the following system.

$$\begin{aligned} 4x &= 0 \\ -2y + 6 &= 0 \end{aligned}$$

This is actually a fairly simple system to solve however. The first equation tells us that $x = 0$ and the second tells us that $y = 3$. So, the only critical point for this function is $(0, 3)$ and this is inside the disk of radius 4. The function value at this critical point is,

$$f(0, 3) = 9$$

Now we need to look at the boundary. This one will be somewhat different from the previous example. In this case we don't have fixed values of x and y on the boundary. Instead we have,

$$x^2 + y^2 = 16$$

We can solve this for x^2 and plug this into the x^2 in $f(x, y)$ to get a function of y as follows.

$$x^2 = 16 - y^2$$

$$g(y) = 2\left(16 - y^2\right) - y^2 + 6y = 32 - 3y^2 + 6y$$

We will need to find the absolute extrema of this function on the range $-4 \le y \le 4$ (this is the range of y's for the disk….). We'll first need the critical points of this function.

$$g'(y) = -6y + 6 \qquad \Rightarrow \qquad y = 1$$

The value of this function at the critical point and the endpoints are,

$$g(-4) = -40 \qquad g(4) = 8 \qquad g(1) = 35$$

Unlike the first example we will still need to find the values of x that correspond to these. We can do this by plugging the value of y into our equation for the circle and solving for x.

$$\begin{aligned} y = -4: &\qquad x^2 = 16 - 16 = 0 \qquad \Rightarrow \qquad x = 0 \\ y = 4: &\qquad x^2 = 16 - 16 = 0 \qquad \Rightarrow \qquad x = 0 \\ y = 1: &\qquad x^2 = 16 - 1 = 15 \qquad \Rightarrow \qquad x = \pm\sqrt{15} = \pm 3.87 \end{aligned}$$

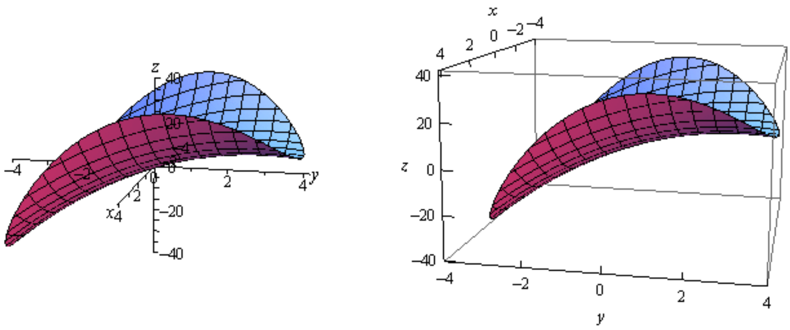
The function values for $g(y)$ then correspond to the following function values for $f(x, y)$.

$$\begin{aligned} g(-4) = -40 &\qquad \Rightarrow \qquad f(0, -4) = -40 \\ g(4) = 8 &\qquad \Rightarrow \qquad f(0, 4) = 8 \\ g(1) = 35 &\qquad \Rightarrow \qquad f\left(-\sqrt{15}, 1\right) = 35 \ \text{ and } \ f\left(\sqrt{15}, 1\right) = 35 \end{aligned}$$

Note that the third one actually corresponded to two different values for $f(x, y)$ since that y also produced two different values of x.

So, comparing these values to the value of the function at the critical point of $f(x, y)$ that we found earlier we can see that the absolute minimum occurs at $(0, -4)$ while the absolute maximum occurs twice at $\left(-\sqrt{15}, 1\right)$ and $\left(\sqrt{15}, 1\right)$.

Here is a sketch of the region for reference purposes.



In both of these examples one of the absolute extrema actually occurred at more than one place. Sometimes this will happen and sometimes it won't so don't read too much into the fact that it happened in both examples given here.

Also note that, as we've seen, absolute extrema will often occur on the boundaries of these regions, although they don't have to occur at the boundaries. Had we given much more complicated examples with multiple critical points we would have seen examples where the absolute extrema occurred interior to the region and not on the boundary.

1. Find the absolute minimum and absolute maximum of $f(x, y) = 192x^3 + y^2 - 4xy^2$ on the triangle with vertices $(0,0)$, $(4,2)$ and $(-2,2)$.

We'll need the first order derivatives to start the problem off. Here they are,

$$f_x = 576x^2 - 4y^2 \qquad f_y = 2y - 8xy$$

We need to find the critical points for this problem. That means solving the following system.

$$\begin{aligned} f_x = 0 &: \quad 576x^2 - 4y^2 = 0 \\ f_y = 0 &: \quad 2y(1 - 4x) = 0 \qquad \rightarrow \qquad y = 0 \text{ or } x = \frac{1}{4} \end{aligned}$$

So, we have two possible options from the second equation. We can plug each into the first equation to get the critical points for the equation.

$$y = 0 : \; 576x^2 = 0 \quad \rightarrow \quad x = 0 \quad \Rightarrow \quad (0,0)$$

$$x = \tfrac{1}{4} : \; 36 - 4y^2 = 0 \quad \rightarrow \quad y = \pm 3 \quad \Rightarrow \quad \left(\tfrac{1}{4}, 3\right) \text{ and } \left(\tfrac{1}{4}, -3\right)$$

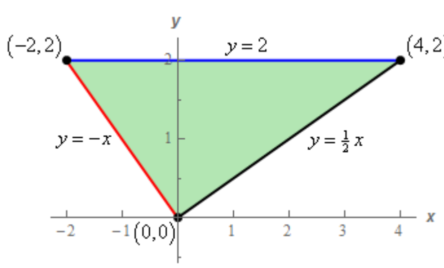
Okay, we have the three critical points listed above. Also recall that we only use critical points that are actually in the region we are working with. In this case, the last two have y values that clearly are out of the region (we've sketched the region in the next step if you aren't sure you believe this!) and so we can ignore them.

Therefore, the only critical point from this list that we need to use is the first. Note as well that, in this case, this also happens to be one of the points that define the boundary of the region. This will happen on occasion but won't always.

So, we'll need the function value for the only critical point that is actually in our region. Here is that value,

$$f(0,0) = 0$$

Now, we know that absolute extrema can occur on the boundary. So, let's start off with a quick sketch of the region we're working on.



Each of the sides of the triangle can then be defined as follows.

Top : $y = 2, \quad -2 \le x \le 4$

Right : $y = \frac{1}{2}x, \quad 0 \le x \le 4$

Left : $y = -x, \quad -2 \le x \le 0$

Now we need to analyze each of these sides to get potential absolute extrema for $f(x, y)$ that might occur on the boundary.

Let's first check out the top :$y = 2, -2 \le x \le 4$.

We'll need to identify the points along the top that could be potential absolute extrema for $f(x, y)$. This, in essence, requires us to find the potential absolute extrema of the following equation on the interval $-2 \le x \le 4$.

The critical point(s) for $g(x)$ are,

$$g'(x) = 576x^2 - 16 = 0 \qquad \rightarrow \qquad x = \pm\frac{1}{6}$$

So, these two points as well as the x limits for the top give the following four points that are potential absolute extrema for $f(x,y)$.

$$\left(\frac{1}{6},2\right) \qquad \left(-\frac{1}{6},2\right) \qquad (-2,2) \qquad (4,2)$$

Recall that, in this step, we are assuming that $y = 2$! So, the next set of potential absolute extrema for $f(x,y)$ are then,

$$f\left(\frac{1}{6},2\right) = \frac{20}{9} \qquad f\left(-\frac{1}{6},2\right) = \frac{52}{9} \qquad f(-2,2) = -1,500 \qquad f(4,2) = 12,228$$

Next let's check out the right side :$y = \frac{1}{2}x,\ \ 0 \le x \le 4$. For this side we'll need to identify possible absolute extrema of the following function on the interval $0 \le x \le 4$.

$$g(x) = f\left(x, \frac{1}{2}x\right) = \frac{1}{4}x^2 + 191x^3$$

The critical point(s) for the $g(x)$ from this step are,

$$g'(x) = \frac{1}{2}x + 573x^2 = x\left(\frac{1}{2} + 573x\right) = 0 \qquad \rightarrow \qquad x = 0, \quad x = -\frac{1}{1146}$$

Now, recall what we are restricted to the interval $0 \le x \le 4$ for this portion of the problem and so the second critical point above will not be used as it lies outside this interval.

So, the single point from above that is in the interval $0 \le x \le 4$ as well as the x limits for the right give the following two points that are potential absolute extrema for $f(x,y)$.

$$(0,0) \qquad (4,2)$$

Recall that, in this step, we are assuming that $y = \frac{1}{2}x$! Also note that, in this case, one of the critical points ended up also being one of the endpoints.

Therefore, the next set of potential absolute extrema for $f(x,y)$ are then,

$$f\left(0,0\right)=0 \qquad f\left(4,2\right)=12,228$$

Before proceeding to the next step note that both of these have already appeared in previous steps. This will happen on occasion but we can't, in many cases, expect this to happen so we do need to go through and do the work for each boundary.

The main exception to this is usually the endpoints of our intervals as they will always be shared in two of the boundary checks and so, once done, don't really need to be checked again. We just included the endpoints here for completeness.

Finally, let's check out the left side :$y=-x,\quad -2\le x\le 0$. For this side we'll need to identify possible absolute extrema of the following function on the interval $-2\le x\le 0$.

$$g\left(x\right)=f\left(x,-x\right)=x^2+188x^3$$

The critical point(s) for the $g\left(x\right)$ from this step are,

$$g'\left(x\right)=2x+564x^2=2x\left(1+282x\right)=0 \qquad \to \qquad x=0,\quad x=-\frac{1}{282}$$

Both of these are in the interval $-2\le x\le 0$ that we are restricted to for this portion of the problem.

So, the two points from above as well as the x limits for the right give the following three points that are potential absolute extrema for $f\left(x,y\right)$.

$$\left(-\frac{1}{282},\frac{1}{282}\right) \qquad \left(0,0\right) \qquad \left(-2,2\right)$$

Recall that, in this step we are assuming that $y=-x$! Also note that, in this case, one of the critical points ended up also being one of the endpoints.

Therefore, the next set of potential absolute extrema for $f\left(x,y\right)$ are then,

$$f\left(-\frac{1}{282},\frac{1}{282}\right)=\frac{1}{238,572} \qquad f\left(0,0\right)=0 \qquad f\left(-2,2\right)=-1,500$$

As with the previous step we can note that both of the end points above have already occurred previously in the problem and didn't really need to be checked here. They were just included for completeness.

Okay, in summary, here are all the potential absolute extrema and their function values for this function on the region we are working on.

$$f\left(\frac{1}{6}, 2\right) = \frac{20}{9} \qquad f\left(-\frac{1}{6}, 2\right) = \frac{52}{9} \qquad f(-2, 2) = -1,500 \qquad f(4, 2) = 12,228$$

$$f(0, 0) = 0 \qquad f\left(-\frac{1}{282}, \frac{1}{282}\right) = \frac{1}{238,572}$$

From this list we can see that the absolute maximum of the function will be 12,228 which occurs at $(4, 2)$ and the absolute minimum of the function will be -1,500 which occurs at $(-2, 2)$.

2. Find the absolute minimum and absolute maximum of $f\left(x,y\right) = \left(9x^2 - 1\right)\left(1 + 4y\right)$ on the rectangle given by $-2 \le x \le 3$, $-1 \le y \le 4$.

We'll need the first order derivatives to start the problem off. Here they are,

$$f_x = 18x\left(1 + 4y\right) \qquad f_y = 4\left(9x^2 - 1\right)$$

We need to find the critical points for this problem. That means solving the following system.

$$\begin{aligned} f_x = 0 \quad &: \quad 18x\left(1 + 4y\right) = 0 \\ f_y = 0 \quad &: \quad 4\left(9x^2 - 1\right) = 0 \qquad \to \qquad x = \pm\frac{1}{3} \end{aligned}$$

So, we have two possible options from the second equation. We can plug each into the first equation to get the critical points for the equation.

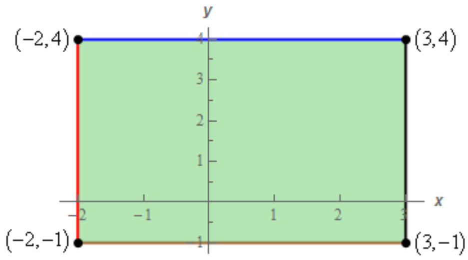
$$x = \frac{1}{3} \;:\; 6\left(1 + 4y\right) = 0 \quad \to \quad y = -\frac{1}{4} \quad \Rightarrow \quad \left(\frac{1}{3}, -\frac{1}{4}\right)$$

$$x = -\frac{1}{3} \;:\; -6\left(1 + 4y\right) = 0 \quad \to \quad y = -\frac{1}{4} \quad \Rightarrow \quad \left(-\frac{1}{3}, -\frac{1}{4}\right)$$

Both of these critical points are in the region we are interested in and so we'll need the function evaluated at both of them. Here are those values,

$$f\left(\frac{1}{3}, -\frac{1}{4}\right) = 0 \qquad f\left(-\frac{1}{3}, -\frac{1}{4}\right) = 0$$

Now, we know that absolute extrema can occur on the boundary. So, let's start off with a quick sketch of the region we're working on.



Each of the sides of the rectangle can then be defined as follows.

Top : $y = 4, \quad -2 \le x \le 3$

Bottom : $y = -1, \quad -2 \le x \le 3$

Right : $x = 3, \quad -1 \le y \le 4$

Left : $x = -2, \quad -1 \le y \le 4$

Now we need to analyze each of these sides to get potential absolute extrema for $f(x, y)$ that might occur on the boundary.

Let's first check out the top : $y = 4, -2 \le x \le 3$.

We'll need to identify the points along the top that could be potential absolute extrema for $f(x, y)$. This, in essence, requires us to find the potential absolute extrema of the following equation on the interval $-2 \le x \le 3$.

The critical point(s) for$g\left(x\right)$ are,

$$g'\left(x\right) = 306x = 0 \qquad \to \qquad x = 0$$

This critical point is in the interval we are working on so, this point as well as the x limits for the top give the following three points that are potential absolute extrema for $f\left(x,y\right)$.

$$\left(0,4\right) \qquad \left(-2,4\right) \qquad \left(3,4\right)$$

Recall that, in this step, we are assuming that $y = 4$! So, the next set of potential absolute extrema for $f\left(x,y\right)$ are then,

$$f\left(0,4\right) = -17 \qquad f\left(-2,4\right) = 595 \qquad f\left(3,4\right) = 1360$$

Next, let's check out the bottom : $y = -1, \quad -2 \le x \le 3$. For this side we'll need to identify possible absolute extrema of the following function on the interval $-2 \le x \le 3$.

$$g\left(x\right) = f\left(x,-1\right) = -3\left(-1 + 9x^2\right)$$

The critical point(s) for the $g\left(x\right)$ from this step are,

$$g'\left(x\right) = -54x = 0 \qquad \to \qquad x = 0$$

This critical point is in the interval we are working on so, this point as well as the x limits for the bottom give the following three points that are potential absolute extrema for $f\left(x,y\right)$.

$$\left(0,-1\right) \qquad \left(-2,-1\right) \qquad \left(3,-1\right)$$

Recall that, in this step, we are assuming that $y = -1$! So, the next set of potential absolute extrema for $f\left(x,y\right)$ are then,

$$f\left(0,-1\right) = 3 \qquad f\left(-2,-1\right) = -105 \qquad f\left(3,-1\right) = -240$$

Let's now check out the right side : $x = 3, \quad -1 \le y \le 4$. For this side we'll need to identify possible absolute extrema of the following function on the interval $-1 \le y \le 4$.

$$h\left(y\right) = f\left(3,y\right) = 80\left(1 + 4y\right)$$

The derivative of the $h\left(y\right)$ from this step is,

$$h'\left(y\right) = 320$$

In this case there are no critical points of the function along this boundary. So, only the limits for the right side are potential absolute extrema for $f\left(x,y\right)$.

$$(3,-1) \qquad (3,4)$$

Recall that, in this step, we are assuming that $x = 3$! Therefore, the next set of potential absolute extrema for $f\left(x,y\right)$ are then,

$$f\left(3,-1\right) = -240 \qquad f\left(3,4\right) = 1360$$

Before proceeding to the next step let's note that both of these points have already been listed in previous steps and so did not really need to be written down here. This will always happen with boundary points (as these are here). Boundary points will always show up in multiple boundary steps.▾

Finally, let's check out the left side : $x = -2, \quad -1 \le y \le 4$. For this side we'll need to identify possible absolute extrema of the following function on the interval $-1 \le y \le 4$.

$$h\left(y\right) = f\left(-2,y\right) = 35\left(1+4y\right)$$

The derivative of the $h\left(y\right)$ from this step is,

$$h'\left(y\right) = 140$$

In this case there are no critical points of the function along this boundary. So, we only the limits for the right side are potential absolute extrema for $f\left(x,y\right)$.

$$(-2,-1) \qquad (-2,4)$$

Recall that, in this step, we are assuming that $x = -2$! Therefore, the next set of potential absolute extrema for $f\left(x,y\right)$ are then,

$$f\left(-2,-1\right) = -105 \qquad f\left(-2,4\right) = 595$$

As with the previous step both of these are boundary points and have appeared in previous steps. They were simply listed here for completeness.

Okay, in summary, here are all the potential absolute extrema and their function values for this function on the region we are working on.

$$\begin{array}{ccc} f\left(0,4\right) = -17 & f\left(-2,4\right) = 595 & f\left(3,4\right) = 1360 \\ f\left(0,-1\right) = 3 & f\left(-2,-1\right) = -105 & f\left(3,-1\right) = -240 \end{array}$$

From this list we can see that the absolute maximum of the function will be 1360 which occurs at $(3,4)$ and the absolute minimum of the function will be -240 which occurs at $(3,-1)$.

0.0.4 Questions Solutions Chain Rule

$$\boxed{7} \qquad \frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}}$$

Again, a version of the **Implicit Function Theorem** stipulates conditions under which our assumption is valid: If F is defined within a sphere containing (a, b, c), where $F(a, b, c) = 0$, $F_z(a, b, c) \neq 0$, and F_x, F_y, and F_z are continuous inside the sphere, then the equation $F(x, y, z) = 0$ defines z as a function of x and y near the point (a, b, c) and this function is differentiable, with partial derivatives given by $\boxed{7}$.

EXAMPLE 9 Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

The solution to Example 9 should be compared to the one in Example 4 in Section 14.3.

SOLUTION Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$. Then, from Equations 7, we have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

14.5 Exercises

1–6 Use the Chain Rule to find dz/dt or ~~dw/dt~~.

1. $z = x^2 + y^2 + xy, \quad x = \sin t, \quad y = e^t$
2. $z = \cos(x + 4y), \quad x = 5t^4, \quad y = 1/t$
3. $z = \sqrt{1 + x^2 + y^2}, \quad x = \ln t, \quad y = \cos t$
4. $z = \tan^{-1}(y/x), \quad x = e^t, \quad y = 1 - e^{-t}$

7–12 Use the Chain Rule to find $\partial z/\partial s$ and $\partial z/\partial t$.

7. $z = x^2y^3, \quad x = s\cos t, \quad y = s\sin t$
8. $z = \arcsin(x - y), \quad x = s^2 + t^2, \quad y = 1 - 2st$
9. $z = \sin\theta\cos\phi, \quad \theta = st^2, \quad \phi = s^2t$
10. $z = e^{x+2y}, \quad x = s/t, \quad y = t/s$
11. $z = e^r\cos\theta, \quad r = st, \quad \theta = \sqrt{s^2 + t^2}$
12. $z = \tan(u/v), \quad u = 2s + 3t, \quad v = 3s - 2t$

13. If $z = f(x, y)$, where f is differentiable, and

$$\begin{array}{ll} x = g(t) & y = h(t) \\ g(3) = 2 & h(3) = 7 \\ g'(3) = 5 & h'(3) = -4 \\ f_x(2, 7) = 6 & f_y(2, 7) = -8 \end{array}$$

find dz/dt when $t = 3$.

1. Homework Hints available at stewartcalculus.com

14.5 The Chain Rule

1. $z = x^2 + y^2 + xy,\ x = \sin t,\ y = e^t \quad\Rightarrow\quad \dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = (2x+y)\cos t + (2y+x)e^t$

2. $z = \cos(x+4y),\ x = 5t^4,\ y = 1/t \quad\Rightarrow$

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = -\sin(x+4y)(1)(20t^3) + [-\sin(x+4y)(4)](-t^{-2}) \\ &= -20t^3\sin(x+4y) + \frac{4}{t^2}\sin(x+4y) = \left(\frac{4}{t^2} - 20t^3\right)\sin(x+4y)\end{aligned}$$

3. $z = \sqrt{1+x^2+y^2},\quad x = \ln t,\quad y = \cos t \quad\Rightarrow$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \tfrac{1}{2}(1+x^2+y^2)^{-1/2}(2x)\cdot\frac{1}{t} + \tfrac{1}{2}(1+x^2+y^2)^{-1/2}(2y)(-\sin t) = \frac{1}{\sqrt{1+x^2+y^2}}\left(\frac{x}{t} - y\sin t\right)$$

4. $z = \tan^{-1}(y/x),\ x = e^t,\ y = 1 - e^{-t} \quad\Rightarrow$

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{1}{1+(y/x)^2}(-yx^{-2})\cdot e^t + \frac{1}{1+(y/x)^2}(1/x)\cdot(-e^{-t})(-1) \\ &= -\frac{y}{x^2+y^2}\cdot e^t + \frac{1}{x+y^2/x}\cdot e^{-t} = \frac{xe^{-t} - ye^t}{x^2+y^2}\end{aligned}$$

5. $w = xe^{y/z},\ x = t^2,\ y = 1-t,\ z = 1+2t \quad\Rightarrow$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = e^{y/z}\cdot 2t + xe^{y/z}\left(\frac{1}{z}\right)\cdot(-1) + xe^{y/z}\left(-\frac{y}{z^2}\right)\cdot 2 = e^{y/z}\left(2t - \frac{x}{z} - \frac{2xy}{z^2}\right)$$

6. $w = \ln\sqrt{x^2+y^2+z^2} = \tfrac{1}{2}\ln(x^2+y^2+z^2),\ x = \sin t,\ y = \cos t,\ z = \tan t \quad\Rightarrow$

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = \frac{1}{2}\cdot\frac{2x}{x^2+y^2+z^2}\cdot\cos t + \frac{1}{2}\cdot\frac{2y}{x^2+y^2+z^2}\cdot(-\sin t) + \frac{1}{2}\cdot\frac{2z}{x^2+y^2+z^2}\cdot\sec^2 t \\ &= \frac{x\cos t - y\sin t + z\sec^2 t}{x^2+y^2+z^2}\end{aligned}$$

7. $z = x^2y^3,\ x = s\cos t,\ y = s\sin t \quad\Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = 2xy^3\cos t + 3x^2y^2\sin t$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (2xy^3)(-s\sin t) + (3x^2y^2)(s\cos t) = -2sxy^3\sin t + 3sx^2y^2\cos t$$

8. $z = \arcsin(x-y),\quad x = s^2+t^2,\quad y = 1-2st \quad\Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \frac{1}{\sqrt{1-(x-y)^2}}(1)\cdot 2s + \frac{1}{\sqrt{1-(x-y)^2}}(-1)\cdot(-2t) = \frac{2s+2t}{\sqrt{1-(x-y)^2}}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{1}{\sqrt{1-(x-y)^2}}(1)\cdot 2t + \frac{1}{\sqrt{1-(x-y)^2}}(-1)\cdot(-2s) = \frac{2s+2t}{\sqrt{1-(x-y)^2}}$$

9. $z = \sin\theta\cos\phi,\quad \theta = st^2,\quad \phi = s^2 t \quad\Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s} + \frac{\partial z}{\partial \phi}\frac{\partial \phi}{\partial s} = (\cos\theta\,\cos\phi)(t^2) + (-\sin\theta\,\sin\phi)(2st) = t^2\cos\theta\,\cos\phi - 2st\sin\theta\,\sin\phi$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t} + \frac{\partial z}{\partial \phi}\frac{\partial \phi}{\partial t} = (\cos\theta\,\cos\phi)(2st) + (-\sin\theta\,\sin\phi)(s^2) = 2st\cos\theta\,\cos\phi - s^2\sin\theta\,\sin\phi$$

10. $z = e^{x+2y},\quad x = s/t,\quad y = t/s \quad\Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (e^{x+2y})(1/t) + (2e^{x+2y})(-ts^{-2}) = e^{x+2y}\left(\frac{1}{t} - \frac{2t}{s^2}\right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (e^{x+2y})(-st^{-2}) + (2e^{x+2y})(1/s) = e^{x+2y}\left(\frac{2}{s} - \frac{s}{t^2}\right)$$

11. $z = e^r\cos\theta,\ r = st,\ \theta = \sqrt{s^2+t^2} \quad\Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s} = e^r\cos\theta\cdot t + e^r(-\sin\theta)\cdot\tfrac{1}{2}(s^2+t^2)^{-1/2}(2s) = te^r\cos\theta - e^r\sin\theta\cdot\frac{s}{\sqrt{s^2+t^2}}$$

$$= e^r\left(t\cos\theta - \frac{s}{\sqrt{s^2+t^2}}\sin\theta\right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t} = e^r\cos\theta\cdot s + e^r(-\sin\theta)\cdot\tfrac{1}{2}(s^2+t^2)^{-1/2}(2t) = se^r\cos\theta - e^r\sin\theta\cdot\frac{t}{\sqrt{s^2+t^2}}$$

$$= e^r\left(s\cos\theta - \frac{t}{\sqrt{s^2+t^2}}\sin\theta\right)$$

12. $z = \tan(u/v),\ u = 2s+3t,\ \ v = 3s-2t \quad\Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial s} = \sec^2(u/v)(1/v)\cdot 2 + \sec^2(u/v)(-uv^{-2})\cdot 3$$

$$= \frac{2}{v}\sec^2\left(\frac{u}{v}\right) - \frac{3u}{v^2}\sec^2\left(\frac{u}{v}\right) = \frac{2v-3u}{v^2}\sec^2\left(\frac{u}{v}\right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial t} = \sec^2(u/v)(1/v)\cdot 3 + \sec^2(u/v)(-uv^{-2})\cdot(-2)$$

$$= \frac{3}{v}\sec^2\left(\frac{u}{v}\right) + \frac{2u}{v^2}\sec^2\left(\frac{u}{v}\right) = \frac{2u+3v}{v^2}\sec^2\left(\frac{u}{v}\right)$$

13. When $t = 3$, $x = g(3) = 2$ and $y = h(3) = 7$. By the Chain Rule (2),

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f_x(2,7)g'(3) + f_y(2,7)\,h'(3) = (6)(5) + (-8)(-4) = 62.$$

14. By the Chain Rule (3), $\dfrac{\partial W}{\partial s} = \dfrac{\partial W}{\partial u}\dfrac{\partial u}{\partial s} + \dfrac{\partial W}{\partial v}\dfrac{\partial v}{\partial s}$. Then

$$W_s(1,0) = F_u(u(1,0),v(1,0))\,u_s(1,0) + F_v(u(1,0),v(1,0))\,v_s(1,0) = F_u(2,3)u_s(1,0) + F_v(2,3)v_s(1,0)$$

$$= (-1)(-2) + (10)(5) = 52$$

Similarly, $\dfrac{\partial W}{\partial t} = \dfrac{\partial W}{\partial u}\dfrac{\partial u}{\partial t} + \dfrac{\partial W}{\partial v}\dfrac{\partial v}{\partial t} \quad\Rightarrow$

$$W_t(1,0) = F_u(u(1,0),v(1,0))\,u_t(1,0) + F_v(u(1,0),v(1,0))\,v_t(1,0) = F_u(2,3)u_t(1,0) + F_v(2,3)v_t(1,0)$$

$$= (-1)(6) + (10)(4) = 34$$

0.0.5 Questions Solutions Directional Derivatives

14.6 Exercises

11–17 Find the directional derivative of the function at the given point in the direction of the vector $\mathbf{v}$.

11. $f(x, y) = e^x \sin y, \quad (0, \pi/3), \quad \mathbf{v} = \langle -6, 8 \rangle$

12. $f(x, y) = \dfrac{x}{x^2 + y^2}, \quad (1, 2), \quad \mathbf{v} = \langle 3, 5 \rangle$

13. $g(p, q) = p^4 - p^2q^3, \quad (2, 1), \quad \mathbf{v} = \mathbf{i} + 3\mathbf{j}$

14. $g(r, s) = \tan^{-1}(rs), \quad (1, 2), \quad \mathbf{v} = 5\mathbf{i} + 10\mathbf{j}$

15. $f(x, y, z) = xe^y + ye^z + ze^x, \quad (0, 0, 0), \quad \mathbf{v} = \langle 5, 1, -2 \rangle$

16. $f(x, y, z) = \sqrt{xyz}, \quad (3, 2, 6), \quad \mathbf{v} = \langle -1, -2, 2 \rangle$

19. Find the directional derivative of $f(x, y) = \sqrt{xy}$ at $P(2, 8)$ in the direction of $Q(5, 4)$.

20. Find the directional derivative of $f(x, y, z) = xy + yz + zx$ at $P(1, -1, 3)$ in the direction of $Q(2, 4, 5)$.

21–26 Find the maximum rate of change of f at the given point and the direction in which it occurs.

21. $f(x, y) = 4y\sqrt{x}, \quad (4, 1)$

22. $f(s, t) = te^{st}, \quad (0, 2)$

23. $f(x, y) = \sin(xy), \quad (1, 0)$

24. $f(x, y, z) = (x + y)/z, \quad (1, 1, -1)$

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11. $f(x,y) = e^x \sin y \quad \Rightarrow \quad \nabla f(x,y) = \langle e^x \sin y, e^x \cos y \rangle$, $\nabla f(0, \pi/3) = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{(-6)^2 + 8^2}} \langle -6, 8 \rangle = \frac{1}{10} \langle -6, 8 \rangle = \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle$, so

$D_{\mathbf{u}}\, f(0, \pi/3) = \nabla f(0, \pi/3) \cdot \mathbf{u} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle \cdot \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle = -\frac{3\sqrt{3}}{10} + \frac{4}{10} = \frac{4 - 3\sqrt{3}}{10}$.

12. $f(x,y) = \dfrac{x}{x^2 + y^2} \quad \Rightarrow \quad \nabla f(x,y) = \left\langle \dfrac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2}, \dfrac{0 - x(2y)}{(x^2+y^2)^2} \right\rangle = \left\langle \dfrac{y^2 - x^2}{(x^2+y^2)^2}, -\dfrac{2xy}{(x^2+y^2)^2} \right\rangle$,

$\nabla f(1,2) = \left\langle \frac{3}{25}, -\frac{4}{25} \right\rangle$, and a unit vector in the direction of $\mathbf{v} = \langle 3, 5 \rangle$ is $\mathbf{u} = \frac{1}{\sqrt{9+25}} \langle 3, 5 \rangle = \left\langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \right\rangle$, so

$D_{\mathbf{u}}\, f(1,2) = \nabla f(1,2) \cdot \mathbf{u} = \left\langle \frac{3}{25}, -\frac{4}{25} \right\rangle \cdot \left\langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \right\rangle = \frac{9}{25\sqrt{34}} - \frac{20}{25\sqrt{34}} = -\frac{11}{25\sqrt{34}}$.

13. $g(p,q) = p^4 - p^2 q^3 \quad \Rightarrow \quad \nabla g(p,q) = \left(4p^3 - 2pq^3\right) \mathbf{i} + \left(-3p^2 q^2\right) \mathbf{j}$, $\nabla g(2,1) = 28\,\mathbf{i} - 12\,\mathbf{j}$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{1^2 + 3^2}} (\mathbf{i} + 3\,\mathbf{j}) = \frac{1}{\sqrt{10}} (\mathbf{i} + 3\,\mathbf{j})$, so

$D_{\mathbf{u}}\, g(2,1) = \nabla g(2,1) \cdot \mathbf{u} = (28\,\mathbf{i} - 12\,\mathbf{j}) \cdot \frac{1}{\sqrt{10}} (\mathbf{i} + 3\,\mathbf{j}) = \frac{1}{\sqrt{10}} (28 - 36) = -\frac{8}{\sqrt{10}}$ or $-\frac{4\sqrt{10}}{5}$.

14. $g(r,s) = \tan^{-1}(rs) \quad \Rightarrow \quad \nabla g(r,s) = \left(\dfrac{1}{1+(rs)^2} \cdot s\right)\mathbf{i} + \left(\dfrac{1}{1+(rs)^2} \cdot r\right)\mathbf{j} = \dfrac{s}{1+r^2s^2}\mathbf{i} + \dfrac{r}{1+r^2s^2}\mathbf{j}$,

$\nabla g(1,2) = \frac{2}{5}\,\mathbf{i} + \frac{1}{5}\,\mathbf{j}$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{5^2+10^2}}(5\,\mathbf{i} + 10\,\mathbf{j}) = \frac{1}{5\sqrt{5}}(5\,\mathbf{i} + 10\,\mathbf{j}) = \frac{1}{\sqrt{5}}\,\mathbf{i} + \frac{2}{\sqrt{5}}\,\mathbf{j}$,

so $D_{\mathbf{u}}\, g(1,2) = \nabla g(1,2) \cdot \mathbf{u} = \left(\frac{2}{5}\,\mathbf{i} + \frac{1}{5}\,\mathbf{j}\right) \cdot \left(\frac{1}{\sqrt{5}}\,\mathbf{i} + \frac{2}{\sqrt{5}}\,\mathbf{j}\right) = \frac{2}{5\sqrt{5}} + \frac{2}{5\sqrt{5}} = \frac{4}{5\sqrt{5}}$ or $\frac{4\sqrt{5}}{25}$.

15. $f(x,y,z) = xe^y + ye^z + ze^x \quad \Rightarrow \quad \nabla f(x,y,z) = \langle e^y + ze^x, xe^y + e^z, ye^z + e^x \rangle$, $\nabla f(0,0,0) = \langle 1,1,1 \rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{25+1+4}}\langle 5,1,-2 \rangle = \frac{1}{\sqrt{30}}\langle 5,1,-2 \rangle$, so

$D_{\mathbf{u}}\, f(0,0,0) = \nabla f(0,0,0) \cdot \mathbf{u} = \langle 1,1,1 \rangle \cdot \frac{1}{\sqrt{30}}\langle 5,1,-2 \rangle = \frac{4}{\sqrt{30}}$.

16. $f(x,y,z) = \sqrt{xyz} \quad \Rightarrow$

$$\nabla f(x,y,z) = \left\langle \tfrac{1}{2}(xyz)^{-1/2} \cdot yz, \tfrac{1}{2}(xyz)^{-1/2} \cdot xz, \tfrac{1}{2}(xyz)^{-1/2} \cdot xy \right\rangle = \left\langle \frac{yz}{2\sqrt{xyz}}, \frac{xz}{2\sqrt{xyz}}, \frac{xy}{2\sqrt{xyz}} \right\rangle,$$

$\nabla f(3,2,6) = \left\langle \frac{12}{2\sqrt{36}}, \frac{18}{2\sqrt{36}}, \frac{6}{2\sqrt{36}} \right\rangle = \left\langle 1, \frac{3}{2}, \frac{1}{2} \right\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{1+4+4}}\langle -1,-2,2 \rangle = \left\langle -\frac{1}{3}, -\frac{2}{3}, \frac{2}{3} \right\rangle$, so

$D_{\mathbf{u}}\, f(3,2,6) = \nabla f(3,2,6) \cdot \mathbf{u} = \left\langle 1, \frac{3}{2}, \frac{1}{2} \right\rangle \cdot \left\langle -\frac{1}{3}, -\frac{2}{3}, \frac{2}{3} \right\rangle = -\frac{1}{3} - 1 + \frac{1}{3} = -1$.

19. $f(x,y) = \sqrt{xy} \quad \Rightarrow \quad \nabla f(x,y) = \left\langle \frac{1}{2}(xy)^{-1/2}(y), \frac{1}{2}(xy)^{-1/2}(x) \right\rangle = \left\langle \dfrac{y}{2\sqrt{xy}}, \dfrac{x}{2\sqrt{xy}} \right\rangle$, so $\nabla f(2,8) = \left\langle 1, \frac{1}{4} \right\rangle$.

The unit vector in the direction of $\overrightarrow{PQ} = \langle 5-2, 4-8 \rangle = \langle 3,-4 \rangle$ is $\mathbf{u} = \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$, so

$D_{\mathbf{u}}\, f(2,8) = \nabla f(2,8) \cdot \mathbf{u} = \left\langle 1, \frac{1}{4} \right\rangle \cdot \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle = \frac{2}{5}$.

20. $f(x,y,z) = xy + yz + zx \quad \Rightarrow \quad \nabla f(x,y,z) = \langle y+z, x+z, y+x \rangle$, so $\nabla f(1,-1,3) = \langle 2,4,0 \rangle$. The unit vector in the direction of $\overrightarrow{PQ} = \langle 1,5,2 \rangle$ is $\mathbf{u} = \frac{1}{\sqrt{30}}\langle 1,5,2 \rangle$, so $D_{\mathbf{u}}\, f(1,-1,3) = \nabla f(1,-1,3) \cdot \mathbf{u} = \langle 2,4,0 \rangle \cdot \frac{1}{\sqrt{30}}\langle 1,5,2 \rangle = \frac{22}{\sqrt{30}}$.

21. $f(x,y) = 4y\sqrt{x} \quad \Rightarrow \quad \nabla f(x,y) = \left\langle 4y \cdot \frac{1}{2}x^{-1/2}, 4\sqrt{x} \right\rangle = \langle 2y/\sqrt{x}, 4\sqrt{x}\,\rangle.$

$\nabla f(4,1) = \langle 1, 8 \rangle$ is the direction of maximum rate of change, and the maximum rate is $|\nabla f(4,1)| = \sqrt{1+64} = \sqrt{65}$.

22. $f(s,t) = te^{st} \quad \Rightarrow \quad \nabla f(s,t) = \langle te^{st}(t), te^{st}(s) + e^{st}(1) \rangle = \langle t^2 e^{st}, (st+1)e^{st} \rangle.$

$\nabla f(0,2) = \langle 4, 1 \rangle$ is the direction of maximum rate of change, and the maximum rate is $|\nabla f(0,2)| = \sqrt{16+1} = \sqrt{17}$.

23. $f(x,y) = \sin(xy) \quad \Rightarrow \quad \nabla f(x,y) = \langle y\cos(xy), x\cos(xy) \rangle$, $\nabla f(1,0) = \langle 0, 1 \rangle$. Thus the maximum rate of change is $|\nabla f(1,0)| = 1$ in the direction $\langle 0, 1 \rangle$.

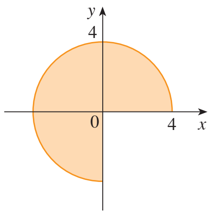
24. $f(x,y,z) = \dfrac{x+y}{z} \quad \Rightarrow \quad \nabla f(x,y,z) = \left\langle \dfrac{1}{z}, \dfrac{1}{z}, -\dfrac{x+y}{z^2} \right\rangle$, $\nabla f(1,1,-1) = \langle -1, -1, -2 \rangle$. Thus the maximum rate of change is $|\nabla f(1,1,-1)| = \sqrt{1+1+4} = \sqrt{6}$ in the direction $\langle -1, -1, -2 \rangle$.

0.0.6 Questions Solutions Double Integral Using POLAR

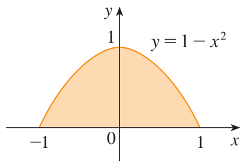
15.4 Exercises

1–4 A region R is shown. Decide whether to use polar coordinates or rectangular coordinates and write $\iint_R f(x, y)\, dA$ as an iterated integral, where f is an arbitrary continuous function on R.

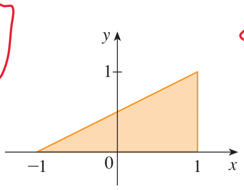
1.



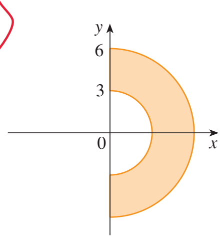
2.



3.



4.



5–6 Sketch the region whose area is given by the integral and evaluate the integral.

5. $\displaystyle\int_{\pi/4}^{3\pi/4} \int_1^2 r\, dr\, d\theta$

6. $\displaystyle\int_{\pi/2}^{\pi} \int_0^{2\sin\theta} r\, dr\, d\theta$

7–14 Evaluate the given integral by changing to polar coordinates.

7. $\iint_D x^2 y\, dA$, where D is the top half of the disk with center the origin and radius 5

8. $\iint_R (2x - y)\, dA$, where R is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $y = x$

9. $\iint_R \sin(x^2 + y^2)\, dA$, where R is the region in the first quadrant between the circles with center the origin and radii 1 and 3

10. $\displaystyle\iint_R \frac{y^2}{x^2 + y^2}\, dA$, where R is the region that lies between the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ with $0 < a < b$

11. $\iint_D e^{-x^2-y^2}\, dA$, where D is the region bounded by the semicircle $x = \sqrt{4 - y^2}$ and the y-axis

12. $\iint_D \cos\sqrt{x^2 + y^2}\, dA$, where D is the disk with center the origin and radius 2

13. $\iint_R \arctan(y/x)\, dA$, where $R = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 4,\ 0 \leq y \leq x\}$

14. $\iint_D x\, dA$, where D is the region in the first quadrant that lies between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 2x$

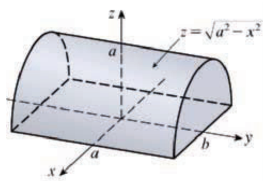
15–18 Use a double integral to find the area of the region.

1. Homework Hints available at stewartcalculus.com

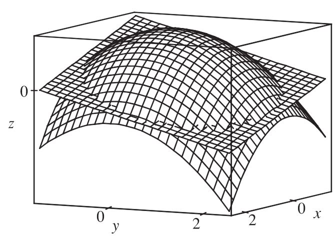
67. $\iint_D \left(ax^3 + by^3 + \sqrt{a^2 - x^2}\,\right) dA = \iint_D ax^3\,dA + \iint_D by^3\,dA + \iint_D \sqrt{a^2 - x^2}\,dA$. Now ax^3 is odd with respect to x and by^3 is odd with respect to y, and the region of integration is symmetric with respect to both x and y, so $\iint_D ax^3\,dA = \iint_D by^3\,dA = 0$.

$\iint_D \sqrt{a^2 - x^2}\,dA$ represents the volume of the solid region under the graph of $z = \sqrt{a^2 - x^2}$ and above the rectangle D, namely a half circular cylinder with radius a and length $2b$ (see the figure) whose volume is $\frac{1}{2} \cdot \pi r^2 h = \frac{1}{2}\pi a^2(2b) = \pi a^2 b$. Thus

$\iint_D \left(ax^3 + by^3 + \sqrt{a^2 - x^2}\,\right) dA = 0 + 0 + \pi a^2 b = \pi a^2 b$.



68. To find the equations of the boundary curves, we require that the z-values of the two surfaces be the same. In Maple, we use the command `solve(4-x^2-y^2=1-x-y,y);` and in Mathematica, we use `Solve[4-x^2-y^2==1-x-y,y]`. We find that the curves have equations $y = \dfrac{1 \pm \sqrt{13 + 4x - 4x^2}}{2}$. To find the two points of intersection of these curves, we use the CAS to solve $13 + 4x - 4x^2 = 0$, finding that $x = \frac{1 \pm \sqrt{14}}{2}$. So, using the CAS to evaluate the integral, the volume of intersection is



$$V = \int_{\left(1 - \sqrt{14}\,\right)/2}^{\left(1 + \sqrt{14}\,\right)/2} \int_{\left(1 - \sqrt{13 + 4x - 4x^2}\,\right)/2}^{\left(1 + \sqrt{13 + 4x - 4x^2}\,\right)/2} \left[(4 - x^2 - y^2) - (1 - x - y)\right] dy\,dx = \frac{49\pi}{8}$$

15.4 Double Integrals in Polar Coordinates

1. The region R is more easily described by polar coordinates: $R = \left\{(r, \theta) \mid 0 \le r \le 4, 0 \le \theta \le \frac{3\pi}{2}\right\}$.

Thus $\iint_R f(x, y)\,dA = \int_0^{3\pi/2} \int_0^4 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

2. The region R is more easily described by rectangular coordinates: $R = \left\{(x, y) \mid -1 \le x \le 1, 0 \le y \le 1 - x^2\right\}$.

Thus $\iint_R f(x, y)\,dA = \int_{-1}^1 \int_0^{1 - x^2} f(x, y)\,dy\,dx$.

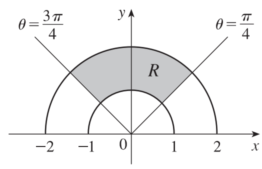
3. The region R is more easily described by rectangular coordinates: $R = \left\{(x, y) \mid -1 \le x \le 1, 0 \le y \le \frac{1}{2}x + \frac{1}{2}\right\}$.

Thus $\iint_R f(x, y)\,dA = \int_{-1}^1 \int_0^{(x+1)/2} f(x, y)\,dy\,dx$.

4. The region R is more easily described by polar coordinates: $R = \left\{(r, \theta) \mid 3 \le r \le 6,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right\}$.

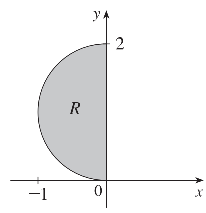
Thus $\iint_R f(x, y)\,dA = \int_{-\pi/2}^{\pi/2} \int_3^6 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

5. The integral $\int_{\pi/4}^{3\pi/4}\int_1^2 r\,dr\,d\theta$ represents the area of the region $R=\{(r,\theta)\mid 1\le r\le 2,\ \pi/4\le\theta\le 3\pi/4\}$, the top quarter portion of a ring (annulus).



$$\begin{aligned}\int_{\pi/4}^{3\pi/4}\int_1^2 r\,dr\,d\theta &= \left(\int_{\pi/4}^{3\pi/4} d\theta\right)\left(\int_1^2 r\,dr\right)\\ &= \Big[\,\theta\,\Big]_{\pi/4}^{3\pi/4}\left[\tfrac12 r^2\right]_1^2 = \left(\tfrac{3\pi}{4}-\tfrac{\pi}{4}\right)\cdot\tfrac12\,(4-1) = \tfrac{\pi}{2}\cdot\tfrac32 = \tfrac{3\pi}{4}\end{aligned}$$

6. The integral $\int_{\pi/2}^{\pi}\int_0^{2\sin\theta} r\,dr\,d\theta$ represents the area of the region $R=\{(r,\theta)\mid 1\le r\le 2\sin\theta,\ \pi/2\le\theta\le\pi\}$. Since $r=2\sin\theta\ \Rightarrow\ r^2=2r\sin\theta\ \Leftrightarrow\ x^2+y^2=2y\ \Leftrightarrow$ $x^2+(y-1)^2=1$, R is the portion in the second quadrant of a disk of radius 1 with center $(0,1)$.



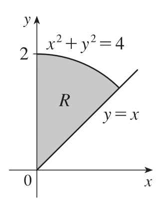
$$\begin{aligned}\int_{\pi/2}^{\pi}\int_0^{2\sin\theta} r\,dr\,d\theta &= \int_{\pi/2}^{\pi}\left[\tfrac12 r^2\right]_{r=0}^{r=2\sin\theta} d\theta = \int_{\pi/2}^{\pi} 2\sin^2\theta\,d\theta\\ &= \int_{\pi/2}^{\pi} 2\cdot\tfrac12(1-\cos 2\theta)\,d\theta = \left[\theta-\tfrac12\sin 2\theta\right]_{\pi/2}^{\pi}\\ &= \pi-0-\tfrac{\pi}{2}+0=\tfrac{\pi}{2}\end{aligned}$$

7. The half disk D can be described in polar coordinates as $D=\{(r,\theta)\mid 0\le r\le 5,\ 0\le\theta\le\pi\}$. Then

$$\begin{aligned}\iint_D x^2y\,dA &= \int_0^{\pi}\int_0^5 (r\cos\theta)^2(r\sin\theta)\,r\,dr\,d\theta = \left(\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta\right)\left(\int_0^5 r^4\,dr\right)\\ &= \left[-\tfrac13\cos^3\theta\right]_0^{\pi}\left[\tfrac15 r^5\right]_0^5 = -\tfrac13(-1-1)\cdot 625 = \tfrac{1250}{3}\end{aligned}$$

8. The region R is $\frac18$ of a disk, as shown in the figure, and can be described by $R=\{(r,\theta)\mid 0\le r\le 2,\ \pi/4\le\theta\le\pi/2\}$. Thus

$$\begin{aligned}\iint_R(2x-y)\,dA &= \int_{\pi/4}^{\pi/2}\int_0^2(2r\cos\theta-r\sin\theta)\,r\,dr\,d\theta\\ &= \left(\int_{\pi/4}^{\pi/2}(2\cos\theta-\sin\theta)\,d\theta\right)\left(\int_0^2 r^2\,dr\right)\\ &= \left[2\sin\theta+\cos\theta\right]_{\pi/4}^{\pi/2}\left[\tfrac13 r^3\right]_0^2\\ &= \left(2+0-\sqrt2-\tfrac{\sqrt2}{2}\right)\left(\tfrac83\right)=\tfrac{16}{3}-4\sqrt2\end{aligned}$$



9. $$\begin{aligned}\iint_R\sin(x^2+y^2)\,dA &= \int_0^{\pi/2}\int_1^3\sin(r^2)\,r\,dr\,d\theta = \left(\int_0^{\pi/2}d\theta\right)\left(\int_1^3 r\sin(r^2)\,dr\right)\\ &= \left[\theta\right]_0^{\pi/2}\left[-\tfrac12\cos(r^2)\right]_1^3\\ &= \left(\tfrac{\pi}{2}\right)\left[-\tfrac12(\cos 9-\cos 1)\right]=\tfrac{\pi}{4}(\cos 1-\cos 9)\end{aligned}$$

10. $$\begin{aligned}\iint_R\frac{y^2}{x^2+y^2}\,dA &= \int_0^{2\pi}\int_a^b\frac{(r\sin\theta)^2}{r^2}\,r\,dr\,d\theta = \left(\int_0^{2\pi}\sin^2\theta\,d\theta\right)\left(\int_a^b r\,dr\right)\\ &= \int_0^{2\pi}\tfrac12(1-\cos 2\theta)\,d\theta\int_a^b r\,dr = \tfrac12\left[\theta-\tfrac12\sin 2\theta\right]_0^{2\pi}\left[\tfrac12 r^2\right]_a^b\\ &= \tfrac12\,(2\pi-0-0)\left[\tfrac12\left(b^2-a^2\right)\right]=\tfrac{\pi}{2}(b^2-a^2)\end{aligned}$$

11. $\iint_D e^{-x^2-y^2}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^2 e^{-r^2}\,r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} d\theta \int_0^2 re^{-r^2}\,dr$

$$= \big[\,\theta\,\big]_{-\pi/2}^{\pi/2}\left[-\tfrac{1}{2}e^{-r^2}\right]_0^2 = \pi\left(-\tfrac{1}{2}\right)(e^{-4}-e^0) = \tfrac{\pi}{2}(1-e^{-4})$$

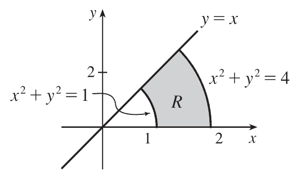
12. $\iint_D \cos\sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^2 \cos\sqrt{r^2}\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^2 r\cos r\,dr$. For the second integral, integrate by parts with $u = r$, $dv = \cos r\,dr$. Then $\iint_D \cos\sqrt{x^2+y^2}\,dA = \big[\,\theta\,\big]_0^{2\pi}\,[r\sin r + \cos r]_0^2 = 2\pi(2\sin 2 + \cos 2 - 1)$.

13. R is the region shown in the figure, and can be described by $R = \{(r,\theta) \mid 0 \le \theta \le \pi/4, 1 \le r \le 2\}$. Thus

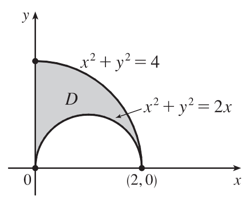
$\iint_R \arctan(y/x)\,dA = \int_0^{\pi/4}\int_1^2 \arctan(\tan\theta)\,r\,dr\,d\theta$ since $y/x = \tan\theta$.

Also, $\arctan(\tan\theta) = \theta$ for $0 \le \theta \le \pi/4$, so the integral becomes

$\int_0^{\pi/4}\int_1^2 \theta\,r\,dr\,d\theta = \int_0^{\pi/4}\theta\,d\theta\int_1^2 r\,dr = \left[\tfrac{1}{2}\theta^2\right]_0^{\pi/4}\left[\tfrac{1}{2}r^2\right]_1^2 = \tfrac{\pi^2}{32}\cdot\tfrac{3}{2} = \tfrac{3}{64}\pi^2.$



14.



$$\iint_D x\,dA = \iint_{\substack{x^2+y^2\le 4\\ x\ge 0,\, y\ge 0}} x\,dA - \iint_{\substack{(x-1)^2+y^2\le 1\\ y\ge 0}} x\,dA$$

$$= \int_0^{\pi/2}\int_0^2 r^2\cos\theta\,dr\,d\theta - \int_0^{\pi/2}\int_0^{2\cos\theta} r^2\cos\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2}\tfrac{1}{3}(8\cos\theta)\,d\theta - \int_0^{\pi/2}\tfrac{1}{3}(8\cos^4\theta)\,d\theta$$

$$= \tfrac{8}{3} - \tfrac{8}{12}\left[\cos^3\theta\sin\theta + \tfrac{3}{2}(\theta + \sin\theta\cos\theta)\right]_0^{\pi/2}$$

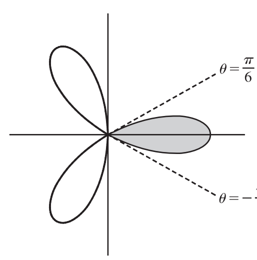
$$= \tfrac{8}{3} - \tfrac{2}{3}\left[0 + \tfrac{3}{2}\left(\tfrac{\pi}{2}\right)\right] = \tfrac{16-3\pi}{6}$$

15. One loop is given by the region $D = \{(r,\theta) \mid -\pi/6 \le \theta \le \pi/6, 0 \le r \le \cos 3\theta\}$, so the area is

$$\iint_D dA = \int_{-\pi/6}^{\pi/6}\int_0^{\cos 3\theta} r\,dr\,d\theta = \int_{-\pi/6}^{\pi/6}\left[\frac{1}{2}r^2\right]_{r=0}^{r=\cos 3\theta} d\theta$$

$$= \int_{-\pi/6}^{\pi/6}\frac{1}{2}\cos^2 3\theta\,d\theta = 2\int_0^{\pi/6}\frac{1}{2}\left(\frac{1+\cos 6\theta}{2}\right)d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{6}\sin 6\theta\right]_0^{\pi/6} = \frac{\pi}{12}$$



16. By symmetry, the area of the region is 4 times the area of the region D in the first quadrant enclosed by the cardiod $r = 1 - \cos\theta$ (see the figure). Here $D = \{(r,\theta) \mid 0 \le r \le 1 - \cos\theta, 0 \le \theta \le \pi/2\}$, so the total area is

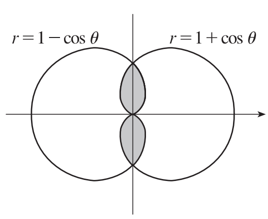
$$4A(D) = 4\iint_D dA = 4\int_0^{\pi/2}\int_0^{1-\cos\theta} r\,dr\,d\theta = 4\int_0^{\pi/2}\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=1-\cos\theta} d\theta$$

$$= 2\int_0^{\pi/2}(1-\cos\theta)^2 d\theta = 2\int_0^{\pi/2}(1 - 2\cos\theta + \cos^2\theta)\,d\theta$$

$$= 2\int_0^{\pi/2}\left[1 - 2\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\right]d\theta$$

$$= 2\left[\theta - 2\sin\theta + \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/2}$$

$$= 2\left(\tfrac{\pi}{2} - 2 + \tfrac{\pi}{4}\right) = \tfrac{3\pi}{2} - 4$$



0.0.7 Questions Solutions MASS Flux

divergence theorem much easier

6. $\iint_S xyz\, dS$,
S is the cone with parametric equations $x = u\cos v$, $y = u\sin v$, $z = u$, $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant \pi/2$

7. $\iint_S y\, dS$, S is the helicoid with vector equation $\mathbf{r}(u, v) = \langle u\cos v, u\sin v, v\rangle$, $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant \pi$

8. $\iint_S (x^2 + y^2)\, dS$,
S is the surface with vector equation $\mathbf{r}(u, v) = \langle 2uv, u^2 - v^2, u^2 + v^2\rangle$, $u^2 + v^2 \leqslant 1$

9. $\iint_S x^2yz\, dS$,
S is the part of the plane $z = 1 + 2x + 3y$ that lies above the rectangle $[0, 3] \times [0, 2]$

10. $\iint_S xz\, dS$,
S is the part of the plane $2x + 2y + z = 4$ that lies in the first octant

11. $\iint_S x\, dS$,
S is the triangular region with vertices $(1, 0, 0)$, $(0, -2, 0)$, and $(0, 0, 4)$

12. $\iint_S y\, dS$,
S is the surface $z = \frac{2}{3}(x^{3/2} + y^{3/2})$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$

13. $\iint_S x^2z^2\, dS$,
S is the part of the cone $z^2 = x^2 + y^2$ that lies between the planes $z = 1$ and $z = 3$

14. $\iint_S z\, dS$,
S is the surface $x = y + 2z^2$, $0 \leqslant y \leqslant 1$, $0 \leqslant z \leqslant 1$

15. $\iint_S y\, dS$,
S is the part of the paraboloid $y = x^2 + z^2$ that lies inside the cylinder $x^2 + z^2 = 4$

16. $\iint_S y^2\, dS$,
S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane

17. $\iint_S (x^2z + y^2z)\, dS$,
S is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geqslant 0$

18. $\iint_S xz\, dS$,
S is the boundary of the region enclosed by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$ and $x + y = 5$

19. $\iint_S (z + x^2y)\, dS$,
S is the part of the cylinder $y^2 + z^2 = 1$ that lies between the planes $x = 0$ and $x = 3$ in the first octant

20. $\iint_S (x^2 + y^2 + z^2)\, dS$,
S is the part of the cylinder $x^2 + y^2 = 9$ between the planes $z = 0$ and $z = 2$, together with its top and bottom disks

21–32 Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for the given vector field $\mathbf{F}$ and the oriented surface S. In other words, find the flux of $\mathbf{F}$ across S. For closed surfaces, use the positive (outward) orientation.

21. $\mathbf{F}(x, y, z) = ze^{xy}\,\mathbf{i} - 3ze^{xy}\,\mathbf{j} + xy\,\mathbf{k}$,
S is the parallelogram of Exercise 5 with upward orientation

22. $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$,
S is the helicoid of Exercise 7 with upward orientation

23. $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$, S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, and has upward orientation

24. $\mathbf{F}(x, y, z) = -x\,\mathbf{i} - y\,\mathbf{j} + z^3\,\mathbf{k}$,
S is the part of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 3$ with downward orientation

25. $\mathbf{F}(x, y, z) = x\,\mathbf{i} - z\,\mathbf{j} + y\,\mathbf{k}$,
S is the part of the sphere $x^2 + y^2 + z^2 = 4$ in the first octant, with orientation toward the origin

26. $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$,
S is the hemisphere $x^2 + y^2 + z^2 = 25$, $y \geqslant 0$, oriented in the direction of the positive y-axis

27. $\mathbf{F}(x, y, z) = y\,\mathbf{j} - z\,\mathbf{k}$,
S consists of the paraboloid $y = x^2 + z^2$, $0 \leqslant y \leqslant 1$, and the disk $x^2 + z^2 \leqslant 1$, $y = 1$

28. $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + 4x^2\,\mathbf{j} + yz\,\mathbf{k}$, S is the surface $z = xe^y$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, with upward orientation

29. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + 2y\,\mathbf{j} + 3z\,\mathbf{k}$,
S is the cube with vertices $(\pm1, \pm1, \pm1)$

30. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + 5\,\mathbf{k}$, S is the boundary of the region enclosed by the cylinder $x^2 + z^2 = 1$ and the planes $y = 0$ and $x + y = 2$

31. $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + y^2\mathbf{j} + z^2\,\mathbf{k}$, S is the boundary of the solid half-cylinder $0 \leqslant z \leqslant \sqrt{1 - y^2}$, $0 \leqslant x \leqslant 2$

32. $\mathbf{F}(x, y, z) = y\,\mathbf{i} + (z - y)\,\mathbf{j} + x\,\mathbf{k}$,
S is the surface of the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$

CAS 33. Evaluate $\iint_S (x^2 + y^2 + z^2)\, dS$ correct to four decimal places, where S is the surface $z = xe^y$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$.

CAS 34. Find the exact value of $\iint_S x^2yz\, dS$, where S is the surface $z = xy$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$.

CAS 35. Find the value of $\iint_S x^2y^2z^2\, dS$ correct to four decimal places, where S is the part of the paraboloid $z = 3 - 2x^2 - y^2$ that lies above the xy-plane.

CAS 36. Find the flux of

$$\mathbf{F}(x, y, z) = \sin(xyz)\,\mathbf{i} + x^2y\,\mathbf{j} + z^2e^{x/5}\,\mathbf{k}$$

across the part of the cylinder $4y^2 + z^2 = 4$ that lies above the xy-plane and between the planes $x = -2$ and $x = 2$ with upward orientation. Illustrate by using a computer algebra system to draw the cylinder and the vector field on the same screen.

37. Find a formula for $\iint_S \mathbf{F} \cdot d\mathbf{S}$ similar to Formula 10 for the case where S is given by $y = h(x, z)$ and $\mathbf{n}$ is the unit normal that points toward the left.

2. Each quarter cylinder has surface area $\frac{1}{4}[2\pi(1)(2)] = \pi$, and the top and bottom disks have surface area $\pi(1)^2 = \pi$. We can

6. $\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u\,\mathbf{k}$, $0 \le u \le 1$, $0 \le v \le \pi/2$ and

$\mathbf{r}_u \times \mathbf{r}_v = (\cos v\,\mathbf{i} + \sin v\,\mathbf{j} + \mathbf{k}) \times (-u\sin v\,\mathbf{i} + u\cos v\,\mathbf{j}) = -u\cos v\,\mathbf{i} - u\sin v\,\mathbf{j} + u\,\mathbf{k} \quad \Rightarrow$

$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2} = \sqrt{2u^2} = \sqrt{2}\,u$ [since $u \ge 0$]. Then by Formula 2,

$$\begin{aligned}\iint_S xyz\,dS &= \iint_D (u\cos v)(u\sin v)(u)\ |\mathbf{r}_u \times \mathbf{r}_v|\ dA = \int_0^1\int_0^{\pi/2}(u^3\sin v\cos v)\cdot\sqrt{2}\,u\,dv\,du \\ &= \sqrt{2}\int_0^1 u^4\,du\int_0^{\pi/2}\sin v\cos v\,dv = \sqrt{2}\left[\tfrac{1}{5}u^5\right]_0^1\ \left[\tfrac{1}{2}\sin^2 v\right]_0^{\pi/2} = \sqrt{2}\cdot\tfrac{1}{5}\cdot\tfrac{1}{2} = \tfrac{1}{10}\sqrt{2}\end{aligned}$$

7. $\mathbf{r}(u, v) = \langle u\cos v, u\sin v, v\rangle$, $0 \le u \le 1$, $0 \le v \le \pi$ and

$\mathbf{r}_u \times \mathbf{r}_v = \langle \cos v, \sin v, 0\rangle \times \langle -u\sin v, u\cos v, 1\rangle = \langle \sin v, -\cos v, u\rangle \quad \Rightarrow$

$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{u^2+1}$. Then

$$\begin{aligned}\iint_S y\,dS &= \iint_D (u\sin v)\ |\mathbf{r}_u \times \mathbf{r}_v|\ dA = \int_0^1\int_0^{\pi}(u\sin v)\cdot\sqrt{u^2+1}\,dv\,du = \int_0^1 u\sqrt{u^2+1}\,du\int_0^{\pi}\sin v\,dv \\ &= \left[\tfrac{1}{3}(u^2+1)^{3/2}\right]_0^1\ \left[-\cos v\right]_0^{\pi} = \tfrac{1}{3}(2^{3/2}-1)\cdot 2 = \tfrac{2}{3}(2\sqrt{2}-1)\end{aligned}$$

8. $\mathbf{r}(u, v) = \left\langle 2uv, u^2 - v^2, u^2 + v^2\right\rangle$, $u^2 + v^2 \le 1$ and

$\mathbf{r}_u \times \mathbf{r}_v = \langle 2v, 2u, 2u\rangle \times \langle 2u, -2v, 2v\rangle = \left\langle 8uv, 4u^2 - 4v^2, -4u^2 - 4v^2\right\rangle$, so

$$\begin{aligned}|\mathbf{r}_u \times \mathbf{r}_v| &= \sqrt{(8uv)^2 + (4u^2-4v^2)^2 + (-4u^2-4v^2)^2} = \sqrt{64u^2v^2 + 32u^4 + 32v^4} \\ &= \sqrt{32(u^2+v^2)^2} = 4\sqrt{2}(u^2+v^2)\end{aligned}$$

Then

$$\iint_S (x^2+y^2)\,dS = \iint_D \left[(2uv)^2 + (u^2-v^2)^2\right] |\mathbf{r}_u \times \mathbf{r}_v|\,dA = \iint_D (4u^2v^2 + u^4 - 2u^2v^2 + v^4)\cdot 4\sqrt{2}(u^2+v^2)\,dA$$

$$= 4\sqrt{2}\iint_D (u^4 + 2u^2v^2 + v^4)(u^2+v^2)\,dA = 4\sqrt{2}\iint_D (u^2+v^2)^3\,dA = 4\sqrt{2}\int_0^{2\pi}\int_0^1 (r^2)^3\,r\,dr\,d\theta$$

$$= 4\sqrt{2}\int_0^{2\pi} d\theta \int_0^1 r^7\,dr = 4\sqrt{2}\,[\theta]_0^{2\pi}\,\left[\tfrac{1}{8}r^8\right]_0^1 = 4\sqrt{2}\cdot 2\pi\cdot\tfrac{1}{8} = \sqrt{2}\,\pi$$

9. $z = 1 + 2x + 3y$ so $\dfrac{\partial z}{\partial x} = 2$ and $\dfrac{\partial z}{\partial y} = 3$. Then by Formula 4,

$$\iint_S x^2yz\,dS = \iint_D x^2yz\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,dA = \int_0^3\int_0^2 x^2y(1+2x+3y)\,\sqrt{4+9+1}\,dy\,dx$$

$$= \sqrt{14}\int_0^3\int_0^2 (x^2y + 2x^3y + 3x^2y^2)\,dy\,dx = \sqrt{14}\int_0^3 \left[\tfrac{1}{2}x^2y^2 + x^3y^2 + x^2y^3\right]_{y=0}^{y=2} dx$$

$$= \sqrt{14}\int_0^3 (10x^2 + 4x^3)\,dx = \sqrt{14}\left[\tfrac{10}{3}x^3 + x^4\right]_0^3 = 171\sqrt{14}$$

10. S is the part of the plane $z = 4 - 2x - 2y$ over the region $D = \{(x,y) \mid 0 \le x \le 2, 0 \le y \le 2 - x\}$. Thus

$$\iint_S xz\,dS = \iint_D x(4-2x-2y)\,\sqrt{(-2)^2 + (-2)^2 + 1}\,dA = 3\int_0^2\int_0^{2-x} (4x - 2x^2 - 2xy)\,dy\,dx$$

$$= 3\int_0^2 \left[4xy - 2x^2y - xy^2\right]_{y=0}^{y=2-x} dx = 3\int_0^2 \left[4x(2-x) - 2x^2(2-x) - x(2-x)^2\right] dx$$

$$= 3\int_0^2 (x^3 - 4x^2 + 4x)\,dx = 3\left[\tfrac{1}{4}x^4 - \tfrac{4}{3}x^3 + 2x^2\right]_0^2 = 3\left(4 - \tfrac{32}{3} + 8\right) = 4$$

11. An equation of the plane through the points $(1,0,0)$, $(0,-2,0)$, and $(0,0,4)$ is $4x - 2y + z = 4$, so S is the region in the plane $z = 4 - 4x + 2y$ over $D = \{(x,y) \mid 0 \le x \le 1, 2x - 2 \le y \le 0\}$. Thus by Formula 4,

$$\iint_S x\,dS = \iint_D x\,\sqrt{(-4)^2 + (2)^2 + 1}\,dA = \sqrt{21}\int_0^1\int_{2x-2}^0 x\,dy\,dx = \sqrt{21}\int_0^1 [xy]_{y=2x-2}^{y=0}\,dx$$

$$= \sqrt{21}\int_0^1 (-2x^2 + 2x)\,dx = \sqrt{21}\left[-\tfrac{2}{3}x^3 + x^2\right]_0^1 = \sqrt{21}\left(-\tfrac{2}{3} + 1\right) = \tfrac{\sqrt{21}}{3}$$

12. $z = \frac{2}{3}(x^{3/2} + y^{3/2})$ and

$$\iint_S y\,dS = \iint_D y\sqrt{(\sqrt{x})^2 + (\sqrt{y})^2 + 1}\,dA = \int_0^1\int_0^1 y\,\sqrt{x+y+1}\,dx\,dy$$

$$= \int_0^1 y\left[\tfrac{2}{3}(x+y+1)^{3/2}\right]_{x=0}^{x=1} dy = \int_0^1 \tfrac{2}{3}y\left[(y+2)^{3/2} - (y+1)^{3/2}\right] dy$$

Substituting $u = y + 2$ in the first term and $t = y + 1$ in the second, we have

$$\iint_S y\,dS = \tfrac{2}{3}\int_2^3 (u-2)u^{3/2}\,du - \tfrac{2}{3}\int_1^2 (t-1)t^{3/2}\,dt = \tfrac{2}{3}\left[\tfrac{2}{7}u^{7/2} - \tfrac{4}{5}u^{5/2}\right]_2^3 - \tfrac{2}{3}\left[\tfrac{2}{7}t^{7/2} - \tfrac{2}{5}t^{5/2}\right]_1^2$$

$$= \tfrac{2}{3}\left[\tfrac{2}{7}(3^{7/2} - 2^{7/2}) - \tfrac{4}{5}(3^{5/2} - 2^{5/2}) - \tfrac{2}{7}(2^{7/2} - 1) + \tfrac{2}{5}(2^{5/2} - 1)\right]$$

$$= \tfrac{2}{3}\left(\tfrac{18}{35}\sqrt{3} + \tfrac{8}{35}\sqrt{2} - \tfrac{4}{35}\right) = \tfrac{4}{105}\left(9\sqrt{3} + 4\sqrt{2} - 2\right)$$

13. S is the portion of the cone $z^2 = x^2 + y^2$ for $1 \le z \le 3$, or equivalently, S is the part of the surface $z = \sqrt{x^2 + y^2}$ over the region $D = \left\{(x,y) \mid 1 \le x^2 + y^2 \le 9\right\}$. Thus

$$\iint_S x^2z^2\,dS = \iint_D x^2(x^2+y^2)\sqrt{\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2+\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2+1}\,dA$$

$$= \iint_D x^2(x^2+y^2)\sqrt{\frac{x^2+y^2}{x^2+y^2}+1}\,dA = \iint_D \sqrt{2}\,x^2(x^2+y^2)\,dA = \sqrt{2}\int_0^{2\pi}\int_1^3 (r\cos\theta)^2(r^2)\,r\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\cos^2\theta\,d\theta\int_1^3 r^5\,dr = \sqrt{2}\left[\tfrac{1}{2}\theta+\tfrac{1}{4}\sin 2\theta\right]_0^{2\pi}\left[\tfrac{1}{6}r^6\right]_1^3 = \sqrt{2}\,(\pi)\cdot\tfrac{1}{6}(3^6-1) = \frac{364\sqrt{2}}{3}\pi$$

14. Using y and z as parameters, we have $\mathbf{r}(y,z) = (y+2z^2)\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, $0 \le y \le 1$, $0 \le z \le 1$.

Then $\mathbf{r}_y \times \mathbf{r}_z = (\mathbf{i}+\mathbf{j}) \times (4z\,\mathbf{i}+\mathbf{k}) = \mathbf{i} - \mathbf{j} - 4z\,\mathbf{k}$ and $|\mathbf{r}_y \times \mathbf{r}_z| = \sqrt{2+16z^2}$. Thus

$$\iint_S z\,dS = \int_0^1\int_0^1 z\sqrt{2+16z^2}\,dy\,dz = \int_0^1 z\sqrt{2+16z^2}\,dz = \left[\tfrac{1}{32}\cdot\tfrac{2}{3}(2+16z^2)^{3/2}\right]_0^1 = \tfrac{1}{48}(18^{3/2}-2^{3/2}) = \tfrac{13}{12}\sqrt{2}.$$

15. Using x and z as parameters, we have $\mathbf{r}(x,z) = x\,\mathbf{i} + (x^2+z^2)\,\mathbf{j} + z\,\mathbf{k}$, $x^2+z^2 \le 4$. Then

$\mathbf{r}_x \times \mathbf{r}_z = (\mathbf{i}+2x\,\mathbf{j}) \times (2z\,\mathbf{j}+\mathbf{k}) = 2x\,\mathbf{i} - \mathbf{j} + 2z\,\mathbf{k}$ and $|\mathbf{r}_x \times \mathbf{r}_z| = \sqrt{4x^2+1+4z^2} = \sqrt{1+4(x^2+z^2)}$. Thus

$$\iint_S y\,dS = \iint_{x^2+z^2\le 4}(x^2+z^2)\sqrt{1+4(x^2+z^2)}\,dA = \int_0^{2\pi}\int_0^2 r^2\sqrt{1+4r^2}\,r\,dr\,d\theta = \int_0^{2\pi}d\theta\int_0^2 r^2\sqrt{1+4r^2}\,r\,dr$$

$$= 2\pi\int_0^2 r^2\sqrt{1+4r^2}\,r\,dr \qquad \left[\text{let } u = 1+4r^2 \;\Rightarrow\; r^2 = \tfrac{1}{4}(u-1) \text{ and } \tfrac{1}{8}du = r\,dr\right]$$

$$= 2\pi\int_1^{17}\tfrac{1}{4}(u-1)\sqrt{u}\cdot\tfrac{1}{8}du = \tfrac{1}{16}\pi\int_1^{17}(u^{3/2}-u^{1/2})\,du$$

$$= \tfrac{1}{16}\pi\left[\tfrac{2}{5}u^{5/2}-\tfrac{2}{3}u^{3/2}\right]_1^{17} = \tfrac{1}{16}\pi\left[\tfrac{2}{5}(17)^{5/2}-\tfrac{2}{3}(17)^{3/2}-\tfrac{2}{5}+\tfrac{2}{3}\right] = \frac{\pi}{60}\left(391\sqrt{17}+1\right)$$

16. The sphere intersects the cylinder in the circle $x^2+y^2=1$, $z=\sqrt{3}$, so S is the portion of the sphere where $z \ge \sqrt{3}$.

Using spherical coordinates to parametrize the sphere we have $\mathbf{r}(\phi,\theta) = 2\sin\phi\,\cos\theta\,\mathbf{i} + 2\sin\phi\,\sin\theta\,\mathbf{j} + 2\cos\phi\,\mathbf{k}$, and

$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = 4\sin\phi$ (see Example 16.6.10). The portion where $z \ge \sqrt{3}$ corresponds to $0 \le \phi \le \frac{\pi}{6}$, $0 \le \theta \le 2\pi$ so

$$\iint_S y^2\,dS = \int_0^{2\pi}\int_0^{\pi/6}(2\sin\phi\sin\theta)^2(4\sin\phi)\,d\phi\,d\theta = 16\int_0^{2\pi}\sin^2\theta\,d\theta\int_0^{\pi/6}\sin^3\phi\,d\phi$$

$$= 16\left[\tfrac{1}{2}\theta-\tfrac{1}{4}\sin 2\theta\right]_0^{2\pi}\left[\tfrac{1}{3}\cos^3\phi-\cos\phi\right]_0^{\pi/6} = 16(\pi)\left(\tfrac{\sqrt{3}}{8}-\tfrac{\sqrt{3}}{2}-\tfrac{1}{3}+1\right) = \left(\tfrac{32}{3}-6\sqrt{3}\right)\pi$$

17. Using spherical coordinates and Example 16.6.10 we have $\mathbf{r}(\phi,\theta) = 2\sin\phi\cos\theta\,\mathbf{i} + 2\sin\phi\sin\theta\,\mathbf{j} + 2\cos\phi\,\mathbf{k}$ and

$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = 4\sin\phi$. Then $\iint_S (x^2z+y^2z)\,dS = \int_0^{2\pi}\int_0^{\pi/2}(4\sin^2\phi)(2\cos\phi)(4\sin\phi)\,d\phi\,d\theta = 16\pi\sin^4\phi\big]_0^{\pi/2} = 16\pi$.

18. Here S consists of three surfaces: S_1, the lateral surface of the cylinder; S_2, the front formed by the plane $x+y=5$;

and the back, S_3, in the plane $x=0$.

On S_1: the surface is given by $\mathbf{r}(u,v) = u\,\mathbf{i} + 3\cos v\,\mathbf{j} + 3\sin v\,\mathbf{k}$, $0 \le v \le 2\pi$, and $0 \le x \le 5-y \;\Rightarrow$

$0 \le u \le 5-3\cos v$. Then $\mathbf{r}_u \times \mathbf{r}_v = -3\cos v\,\mathbf{j} - 3\sin v\,\mathbf{k}$ and $|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{9\cos^2 v + 9\sin^2 v} = 3$, so

$$\iint_{S_1} xz\,dS = \int_0^{2\pi}\int_0^{5-3\cos v} u(3\sin v)(3)\,du\,dv = 9\int_0^{2\pi}\left[\tfrac{1}{2}u^2\right]_{u=0}^{u=5-3\cos v}\sin v\,dv$$

$$= \tfrac{9}{2}\int_0^{2\pi}(5-3\cos v)^2\sin v\,dv = \tfrac{9}{2}\left[\tfrac{1}{9}(5-3\cos v)^3\right]_0^{2\pi} = 0.$$

On S_2: $\mathbf{r}(y,z) = (5-y)\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $|\mathbf{r}_y \times \mathbf{r}_z| = |\mathbf{i}+\mathbf{j}| = \sqrt{2}$, where $y^2+z^2 \le 9$ and

$$\begin{aligned}\iint_{S_2} xz\,dS &= \iint_{y^2+z^2\le 9} (5-y)z\,\sqrt{2}\,dA = \sqrt{2}\int_0^{2\pi}\int_0^3 (5-r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta \\ &= \sqrt{2}\int_0^{2\pi}\int_0^3 (5r^2 - r^3\cos\theta)(\sin\theta)\,dr\,d\theta = \sqrt{2}\int_0^{2\pi}\left[\tfrac{5}{3}r^3 - \tfrac{1}{4}r^4\cos\theta\right]_{r=0}^{r=3}\sin\theta\,d\theta \\ &= \sqrt{2}\int_0^{2\pi}\left(45 - \tfrac{81}{4}\cos\theta\right)\sin\theta\,d\theta = \sqrt{2}\left(\tfrac{4}{81}\right)\cdot\tfrac{1}{2}\left(45-\tfrac{81}{4}\cos\theta\right)^2\Big]_0^{2\pi} = 0\end{aligned}$$

On S_3: $x = 0$ so $\iint_{S_3} xz\,dS = 0$. Hence $\iint_S xz\,dS = 0+0+0 = 0$.

19. S is given by $\mathbf{r}(u,v) = u\,\mathbf{i} + \cos v\,\mathbf{j} + \sin v\,\mathbf{k}$, $0 \le u \le 3$, $0 \le v \le \pi/2$. Then

$\mathbf{r}_u \times \mathbf{r}_v = \mathbf{i} \times (-\sin v\,\mathbf{j} + \cos v\,\mathbf{k}) = -\cos v\,\mathbf{j} - \sin v\,\mathbf{k}$ and $|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{\cos^2 v + \sin^2 v} = 1$, so

$$\begin{aligned}\iint_S (z + x^2 y)\,dS &= \int_0^{\pi/2}\int_0^3 (\sin v + u^2\cos v)(1)\,du\,dv = \int_0^{\pi/2}(3\sin v + 9\cos v)\,dv \\ &= [-3\cos v + 9\sin v]_0^{\pi/2} = 0 + 9 + 3 - 0 = 12\end{aligned}$$

20. Let S_1 be the lateral surface, S_2 the top disk, and S_3 the bottom disk.

On S_1: $\mathbf{r}(\theta,z) = 3\cos\theta\,\mathbf{i} + 3\sin\theta\,\mathbf{j} + z\,\mathbf{k}$, $0 \le \theta \le 2\pi$, $0 \le z \le 2$, $|\mathbf{r}_\theta \times \mathbf{r}_z| = 3$,

$\iint_{S_1} (x^2+y^2+z^2)\,dS = \int_0^{2\pi}\int_0^2 (9+z^2)\,3\,dz\,d\theta = 2\pi(54+8) = 124\pi$.

On S_2: $\mathbf{r}(\theta,r) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + 2\,\mathbf{k}$, $0 \le r \le 3$, $0 \le \theta \le 2\pi$, $|\mathbf{r}_\theta \times \mathbf{r}_r| = r$,

$\iint_{S_2} (x^2+y^2+z^2)\,dS = \int_0^{2\pi}\int_0^3 (r^2+4)\,r\,dr\,d\theta = 2\pi\left(\frac{81}{4}+18\right) = \frac{153}{2}\pi$.

On S_3: $\mathbf{r}(\theta,r) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}$, $0 \le r \le 3$, $0 \le \theta \le 2\pi$, $|\mathbf{r}_\theta \times \mathbf{r}_r| = r$,

$\iint_{S_3} (x^2+y^2+z^2)\,dS = \int_0^{2\pi}\int_0^3 (r^2+0)\,r\,dr\,d\theta = 2\pi\left(\frac{81}{4}\right) = \frac{81}{2}\pi$.

Hence $\iint_S \left(x^2+y^2+z^2\right)dS = 124\pi + \frac{153}{2}\pi + \frac{81}{2}\pi = 241\pi$.

21. From Exercise 5, $\mathbf{r}(u,v) = (u+v)\,\mathbf{i} + (u-v)\,\mathbf{j} + (1+2u+v)\,\mathbf{k}$, $0 \le u \le 2$, $0 \le v \le 1$, and $\mathbf{r}_u \times \mathbf{r}_v = 3\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$. Then

$$\begin{aligned}\mathbf{F}(\mathbf{r}(u,v)) &= (1+2u+v)e^{(u+v)(u-v)}\,\mathbf{i} - 3(1+2u+v)e^{(u+v)(u-v)}\,\mathbf{j} + (u+v)(u-v)\,\mathbf{k} \\ &= (1+2u+v)e^{u^2-v^2}\,\mathbf{i} - 3(1+2u+v)e^{u^2-v^2}\,\mathbf{j} + (u^2-v^2)\,\mathbf{k}\end{aligned}$$

Because the z-component of $\mathbf{r}_u \times \mathbf{r}_v$ is negative we use $-(\mathbf{r}_u \times \mathbf{r}_v)$ in Formula 9 for the upward orientation:

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D \mathbf{F}\cdot(-(\mathbf{r}_u \times \mathbf{r}_v))\,dA = \int_0^1\int_0^2 \left[-3(1+2u+v)e^{u^2-v^2} + 3(1+2u+v)e^{u^2-v^2} + 2(u^2-v^2)\right]du\,dv \\ &= \int_0^1\int_0^2 2(u^2-v^2)\,du\,dv = 2\int_0^1\left[\tfrac{1}{3}u^3 - uv^2\right]_{u=0}^{u=2}dv = 2\int_0^1\left(\tfrac{8}{3} - 2v^2\right)dv \\ &= 2\left[\tfrac{8}{3}v - \tfrac{2}{3}v^3\right]_0^1 = 2\left(\tfrac{8}{3} - \tfrac{2}{3}\right) = 4\end{aligned}$$

22. $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, v\rangle$, $0 \le u \le 1$, $0 \le v \le \pi$ and

$\mathbf{r}_u \times \mathbf{r}_v = \langle \cos v, \sin v, 0\rangle \times \langle -u\sin v, u\cos v, 1\rangle = \langle \sin v, -\cos v, u\rangle$. Here $\mathbf{F}(\mathbf{r}(u,v)) = v\,\mathbf{i} + u\sin v\,\mathbf{j} + u\cos v\,\mathbf{k}$ and, by Formula 9,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,dA = \int_0^1\int_0^\pi (v\sin v - u\sin v\cos v + u^2\cos v)\,dv\,du$$
$$= \int_0^1 \left[\sin v - v\cos v - \tfrac{1}{2}u\sin^2 v + u^2\sin v\right]_{v=0}^{v=\pi} du = \int_0^1 \pi\,du = \pi u\big]_0^1 = \pi$$

23. $\mathbf{F}(x,y,z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$, $z = g(x,y) = 4 - x^2 - y^2$, and D is the square $[0,1]\times[0,1]$, so by Equation 10

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D[-xy(-2x) - yz(-2y) + zx]\,dA = \int_0^1\int_0^1[2x^2y + 2y^2(4-x^2-y^2) + x(4-x^2-y^2)]\,dy\,dx$$
$$= \int_0^1\left(\tfrac{1}{3}x^2 + \tfrac{11}{3}x - x^3 + \tfrac{34}{15}\right)dx = \tfrac{713}{180}$$

24. $\mathbf{F}(x,y,z) = -x\,\mathbf{i} - y\,\mathbf{j} + z^3\,\mathbf{k}$, $z = g(x,y) = \sqrt{x^2+y^2}$, and D is the annular region $\{(x,y) \mid 1 \le x^2+y^2 \le 9\}$. Since S has downward orientation, we have

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -\iint_D\left[-(-x)\left(\frac{x}{\sqrt{x^2+y^2}}\right) - (-y)\left(\frac{y}{\sqrt{x^2+y^2}}\right) + z^3\right]dA$$
$$= -\iint_D\left[\frac{x^2+y^2}{\sqrt{x^2+y^2}} + \left(\sqrt{x^2+y^2}\right)^3\right]dA = -\int_0^{2\pi}\int_1^3\left(\frac{r^2}{r} + r^3\right)r\,dr\,d\theta$$
$$= -\int_0^{2\pi}d\theta\int_1^3(r^2+r^4)\,dr = -\left[\theta\right]_0^{2\pi}\left[\tfrac{1}{3}r^3 + \tfrac{1}{5}r^5\right]_1^3$$
$$= -2\pi\left(9 + \tfrac{243}{5} - \tfrac{1}{3} - \tfrac{1}{5}\right) = -\tfrac{1712}{15}\pi$$

25. $\mathbf{F}(x,y,z) = x\,\mathbf{i} - z\,\mathbf{j} + y\,\mathbf{k}$, $z = g(x,y) = \sqrt{4-x^2-y^2}$ and D is the quarter disk $\left\{(x,y) \,\middle|\, 0\le x\le 2, 0\le y\le\sqrt{4-x^2}\right\}$. S has downward orientation, so by Formula 10,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -\iint_D\left[-x\cdot\tfrac{1}{2}(4-x^2-y^2)^{-1/2}(-2x) - (-z)\cdot\tfrac{1}{2}(4-x^2-y^2)^{-1/2}(-2y) + y\right]dA$$
$$= -\iint_D\left(\frac{x^2}{\sqrt{4-x^2-y^2}} - \sqrt{4-x^2-y^2}\cdot\frac{y}{\sqrt{4-x^2-y^2}} + y\right)dA$$
$$= -\iint_D x^2(4-(x^2+y^2))^{-1/2}\,dA = -\int_0^{\pi/2}\int_0^2(r\cos\theta)^2(4-r^2)^{-1/2}\,r\,dr\,d\theta$$
$$= -\int_0^{\pi/2}\cos^2\theta\,d\theta\int_0^2 r^3(4-r^2)^{-1/2}\,dr \qquad \left[\text{let } u = 4-r^2 \;\Rightarrow\; r^2 = 4-u \text{ and } -\tfrac{1}{2}\,du = r\,dr\right]$$
$$= -\int_0^{\pi/2}\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\right)d\theta\int_4^0 -\tfrac{1}{2}(4-u)(u)^{-1/2}\,du$$
$$= -\left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/2}\left(-\tfrac{1}{2}\right)\left[8\sqrt{u} - \tfrac{2}{3}u^{3/2}\right]_4^0 = -\tfrac{\pi}{4}\left(-\tfrac{1}{2}\right)\left(-16 + \tfrac{16}{3}\right) = -\tfrac{4}{3}\pi$$

26. $\mathbf{F}(x,y,z) = xz\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$

Using spherical coordinates, S is given by $x = 5\sin\phi\cos\theta$, $y = 5\sin\phi\sin\theta$, $z = 5\cos\phi$, $0\le\theta\le\pi$, $0\le\phi\le\pi$. $\mathbf{F}(\mathbf{r}(\phi,\theta)) = (5\sin\phi\cos\theta)(5\cos\phi)\,\mathbf{i} + (5\sin\phi\cos\theta)\,\mathbf{j} + (5\sin\phi\sin\theta)\,\mathbf{k}$ and $\mathbf{r}_\phi\times\mathbf{r}_\theta = 25\sin^2\phi\cos\theta\,\mathbf{i} + 25\sin^2\phi\sin\theta\,\mathbf{j} + 25\cos\phi\sin\phi\,\mathbf{k}$, so

$$\mathbf{F}(\mathbf{r}(\phi,\theta))\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta) = 625\sin^3\phi\cos\phi\cos^2\theta + 125\sin^3\phi\cos\theta\sin\theta + 125\sin^2\phi\cos\phi\sin\theta$$

Then

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D [\mathbf{F}(\mathbf{r}(\phi,\theta))\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta)]\,dA\\ &= \int_0^\pi\int_0^\pi (625\sin^3\phi\cos\phi\cos^2\theta + 125\sin^3\phi\cos\theta\sin\theta + 125\sin^2\phi\cos\phi\sin\theta)\,d\theta\,d\phi\\ &= 125\int_0^\pi \left[5\sin^3\phi\cos\phi\left(\tfrac{1}{2}\theta+\tfrac{1}{4}\sin 2\theta\right)+\sin^3\phi\left(\tfrac{1}{2}\sin^2\theta\right)+\sin^2\phi\cos\phi\,(-\cos\theta)\right]_{\theta=0}^{\theta=\pi}d\phi\\ &= 125\int_0^\pi\left(\tfrac{5}{2}\pi\sin^3\phi\cos\phi + 2\sin^2\phi\cos\phi\right)d\phi = 125\left[\tfrac{5}{2}\pi\cdot\tfrac{1}{4}\sin^4\phi + 2\cdot\tfrac{1}{3}\sin^3\phi\right]_0^\pi = 0\end{aligned}$$

27. Let S_1 be the paraboloid $y = x^2+z^2$, $0\le y\le 1$ and S_2 the disk $x^2+z^2\le 1$, $y=1$. Since S is a closed surface, we use the outward orientation.

On S_1: $\mathbf{F}(\mathbf{r}(x,z)) = (x^2+z^2)\,\mathbf{j} - z\,\mathbf{k}$ and $\mathbf{r}_x\times\mathbf{r}_z = 2x\,\mathbf{i}-\mathbf{j}+2z\,\mathbf{k}$ (since the $\mathbf{j}$-component must be negative on S_1). Then

$$\begin{aligned}\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} &= \iint_{x^2+z^2\le 1}[-(x^2+z^2)-2z^2]\,dA = -\int_0^{2\pi}\int_0^1(r^2+2r^2\sin^2\theta)\,r\,dr\,d\theta\\ &= -\int_0^{2\pi}\int_0^1 r^3(1+2\sin^2\theta)\,dr\,d\theta = -\int_0^{2\pi}(1+1-\cos 2\theta)\,d\theta\int_0^1 r^3\,dr\\ &= -\left[2\theta-\tfrac{1}{2}\sin 2\theta\right]_0^{2\pi}\left[\tfrac{1}{4}r^4\right]_0^1 = -4\pi\cdot\tfrac{1}{4} = -\pi\end{aligned}$$

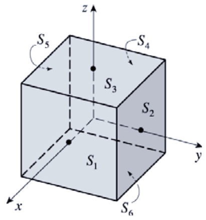
On S_2: $\mathbf{F}(\mathbf{r}(x,z)) = \mathbf{j}-z\,\mathbf{k}$ and $\mathbf{r}_z\times\mathbf{r}_x = \mathbf{j}$. Then $\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \iint_{x^2+z^2\le 1}(1)\,dA = \pi$.

Hence $\iint_S\mathbf{F}\cdot d\mathbf{S} = -\pi+\pi = 0$.

28. $\mathbf{F}(x,y,z) = xy\,\mathbf{i}+4x^2\,\mathbf{j}+yz\,\mathbf{k}$, $z = g(x,y) = xe^y$, and D is the square $[0,1]\times[0,1]$, so by Equation 10

$$\begin{aligned}\iint_S\mathbf{F}\cdot d\mathbf{S} &= \iint_D[-xy(e^y)-4x^2(xe^y)+yz]dA = \int_0^1\int_0^1(-xye^y-4x^3e^y+xye^y)\,dy\,dx\\ &= \int_0^1\left[-4x^3e^y\right]_{y=0}^{y=1}dx = (e-1)\int_0^1(-4x^3)\,dx = 1-e\end{aligned}$$

29. Here S consists of the six faces of the cube as labeled in the figure. On S_1:



$\mathbf{F} = \mathbf{i}+2y\,\mathbf{j}+3z\,\mathbf{k}$, $\mathbf{r}_y\times\mathbf{r}_z = \mathbf{i}$ and $\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \int_{-1}^1\int_{-1}^1 dy\,dz = 4$;

S_2: $\mathbf{F} = x\,\mathbf{i}+2\,\mathbf{j}+3z\,\mathbf{k}$, $\mathbf{r}_z\times\mathbf{r}_x = \mathbf{j}$ and $\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \int_{-1}^1\int_{-1}^1 2\,dx\,dz = 8$;

S_3: $\mathbf{F} = x\,\mathbf{i}+2y\,\mathbf{j}+3\,\mathbf{k}$, $\mathbf{r}_x\times\mathbf{r}_y = \mathbf{k}$ and $\iint_{S_3}\mathbf{F}\cdot d\mathbf{S} = \int_{-1}^1\int_{-1}^1 3\,dx\,dy = 12$;

S_4: $\mathbf{F} = -\mathbf{i}+2y\,\mathbf{j}+3z\,\mathbf{k}$, $\mathbf{r}_z\times\mathbf{r}_y = -\mathbf{i}$ and $\iint_{S_4}\mathbf{F}\cdot d\mathbf{S} = 4$;

S_5: $\mathbf{F} = x\,\mathbf{i}-2\,\mathbf{j}+3z\,\mathbf{k}$, $\mathbf{r}_x\times\mathbf{r}_z = -\mathbf{j}$ and $\iint_{S_5}\mathbf{F}\cdot d\mathbf{S} = 8$;

S_6: $\mathbf{F} = x\,\mathbf{i}+2y\,\mathbf{j}-3\,\mathbf{k}$, $\mathbf{r}_y\times\mathbf{r}_x = -\mathbf{k}$ and $\iint_{S_6}\mathbf{F}\cdot d\mathbf{S} = \int_{-1}^1\int_{-1}^1 3\,dx\,dy = 12$.

Hence $\iint_S\mathbf{F}\cdot d\mathbf{S} = \sum_{i=1}^6\iint_{S_i}\mathbf{F}\cdot d\mathbf{S} = 48$.

30. Here S consists of three surfaces: S_1, the lateral surface of the cylinder; S_2, the front formed by the plane $x+y=2$; and the back, S_3, in the plane $y=0$.

On S_1: $\mathbf{F}(\mathbf{r}(\theta,y)) = \sin\theta\,\mathbf{i}+y\,\mathbf{j}+5\,\mathbf{k}$ and $\mathbf{r}_\theta\times\mathbf{r}_y = \sin\theta\,\mathbf{i}+\cos\theta\,\mathbf{k}$ $\Rightarrow$

$$\begin{aligned}\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^{2-\sin\theta}(\sin^2\theta+5\cos\theta)\,dy\,d\theta\\ &= \int_0^{2\pi}(2\sin^2\theta+10\cos\theta-\sin^3\theta-5\sin\theta\,\cos\theta)\,d\theta = 2\pi\end{aligned}$$

0.0.8 Questions Solutions Fundamental Theorem line INTEGRAL

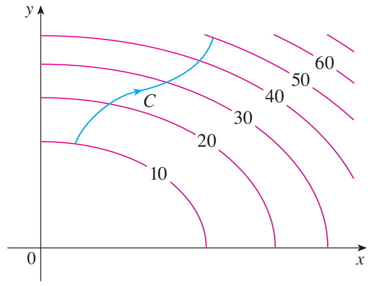
Comparing this equation with Equation 16, we see that

$$P(A) + K(A) = P(B) + K(B)$$

which says that if an object moves from one point A to another point B under the influence of a conservative force field, then the sum of its potential energy and its kinetic energy remains constant. This is called the **Law of Conservation of Energy** and it is the reason the vector field is called *conservative*.

16.3 Exercises

1. The figure shows a curve C and a contour map of a function f whose gradient is continuous. Find $\int_C \nabla f \cdot d\mathbf{r}$.



2. A table of values of a function f with continuous gradient is given. Find $\int_C \nabla f \cdot d\mathbf{r}$, where C has parametric equations

$$x = t^2 + 1 \qquad y = t^3 + t \qquad 0 \leqslant t \leqslant 1$$

$x \backslash y$	0	1	2
0	1	6	4
1	3	5	7
2	8	2	9

3–10 Determine whether or not $\mathbf{F}$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

3. $\mathbf{F}(x, y) = (2x - 3y)\,\mathbf{i} + (-3x + 4y - 8)\,\mathbf{j}$

4. $\mathbf{F}(x, y) = e^x \sin y\,\mathbf{i} + e^x \cos y\,\mathbf{j}$

5. $\mathbf{F}(x, y) = e^x \cos y\,\mathbf{i} + e^x \sin y\,\mathbf{j}$

6. $\mathbf{F}(x, y) = (3x^2 - 2y^2)\,\mathbf{i} + (4xy + 3)\,\mathbf{j}$

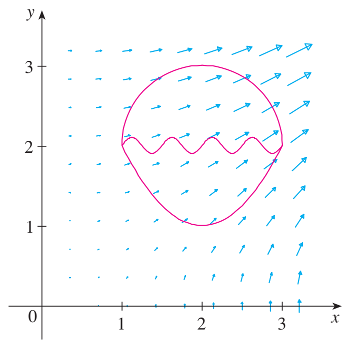
7. $\mathbf{F}(x, y) = (ye^x + \sin y)\,\mathbf{i} + (e^x + x\cos y)\,\mathbf{j}$

8. $\mathbf{F}(x, y) = (2xy + y^{-2})\,\mathbf{i} + (x^2 - 2xy^{-3})\,\mathbf{j}, \quad y > 0$

9. $\mathbf{F}(x, y) = (\ln y + 2xy^3)\,\mathbf{i} + (3x^2y^2 + x/y)\,\mathbf{j}$

10. $\mathbf{F}(x, y) = (xy\cosh xy + \sinh xy)\,\mathbf{i} + (x^2\cosh xy)\,\mathbf{j}$

11. The figure shows the vector field $\mathbf{F}(x, y) = \langle 2xy, x^2 \rangle$ and three curves that start at (1, 2) and end at (3, 2).
(a) Explain why $\int_C \mathbf{F} \cdot d\mathbf{r}$ has the same value for all three curves.
(b) What is this common value?



12–18 (a) Find a function f such that $\mathbf{F} = \nabla f$ and (b) use part (a) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

12. $\mathbf{F}(x, y) = x^2\,\mathbf{i} + y^2\,\mathbf{j}$,
C is the arc of the parabola $y = 2x^2$ from $(-1, 2)$ to $(2, 8)$

13. $\mathbf{F}(x, y) = xy^2\,\mathbf{i} + x^2y\,\mathbf{j}$,
C: $\mathbf{r}(t) = \left\langle t + \sin\tfrac{1}{2}\pi t,\ t + \cos\tfrac{1}{2}\pi t \right\rangle, \quad 0 \leqslant t \leqslant 1$

14. $\mathbf{F}(x, y) = (1 + xy)e^{xy}\,\mathbf{i} + x^2e^{xy}\,\mathbf{j}$,
C: $\mathbf{r}(t) = \cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}, \quad 0 \leqslant t \leqslant \pi/2$

15. $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + xz\,\mathbf{j} + (xy + 2z)\,\mathbf{k}$,
C is the line segment from $(1, 0, -2)$ to $(4, 6, 3)$

CAS Computer algebra system required **1.** Homework Hints available at stewartcalculus.com

16. $\mathbf{F}(x, y, z) = (y^2z + 2xz^2)\,\mathbf{i} + 2xyz\,\mathbf{j} + (xy^2 + 2x^2z)\,\mathbf{k}$,
C: $x = \sqrt{t},\ y = t + 1,\ z = t^2, \quad 0 \leqslant t \leqslant 1$

17. $\mathbf{F}(x, y, z) = yze^{xz}\,\mathbf{i} + e^{xz}\,\mathbf{j} + xye^{xz}\,\mathbf{k}$,
C: $\mathbf{r}(t) = (t^2 + 1)\,\mathbf{i} + (t^2 - 1)\,\mathbf{j} + (t^2 - 2t)\,\mathbf{k}, \quad 0 \leqslant t \leqslant 2$

18. $\mathbf{F}(x, y, z) = \sin y\,\mathbf{i} + (x\cos y + \cos z)\,\mathbf{j} - y\sin z\,\mathbf{k}$,
C: $\mathbf{r}(t) = \sin t\,\mathbf{i} + t\,\mathbf{j} + 2t\,\mathbf{k}, \quad 0 \leqslant t \leqslant \pi/2$

19–20 Show that the line integral is independent of path and evaluate the integral.

19. $\int_C 2xe^{-y}\,dx + (2y - x^2e^{-y})\,dy$,
C is any path from $(1, 0)$ to $(2, 1)$

20. $\int_C \sin y\,dx + (x\cos y - \sin y)\,dy$,
C is any path from $(2, 0)$ to $(1, \pi)$

21. Suppose you're asked to determine the curve that requires the least work for a force field $\mathbf{F}$ to move a particle from one point to another point. You decide to check first whether $\mathbf{F}$ is conservative, and indeed it turns out that it is. How would you reply to the request?

22. Suppose an experiment determines that the amount of work required for a force field $\mathbf{F}$ to move a particle from the point $(1, 2)$ to the point $(5, -3)$ along a curve C_1 is 1.2 J and the work done by $\mathbf{F}$ in moving the particle along another curve C_2 between the same two points is 1.4 J. What can you say about $\mathbf{F}$? Why?

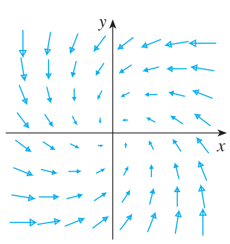
23–24 Find the work done by the force field $\mathbf{F}$ in moving an object from P to Q.

23. $\mathbf{F}(x, y) = 2y^{3/2}\,\mathbf{i} + 3x\sqrt{y}\,\mathbf{j}$; $\quad P(1, 1),\ Q(2, 4)$

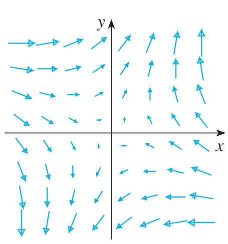
24. $\mathbf{F}(x, y) = e^{-y}\,\mathbf{i} - xe^{-y}\,\mathbf{j}$; $\quad P(0, 1),\ Q(2, 0)$

25–26 Is the vector field shown in the figure conservative? Explain.

25.



26.



CAS **27.** If $\mathbf{F}(x, y) = \sin y\,\mathbf{i} + (1 + x\cos y)\,\mathbf{j}$, use a plot to guess whether $\mathbf{F}$ is conservative. Then determine whether your guess is correct.

28. Let $\mathbf{F} = \nabla f$, where $f(x, y) = \sin(x - 2y)$. Find curves C_1 and C_2 that are not closed and satisfy the equation.

(a) $\displaystyle\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0$ $\qquad$ (b) $\displaystyle\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 1$

29. Show that if the vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is conservative and P, Q, R have continuous first-order partial derivatives, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

30. Use Exercise 29 to show that the line integral $\int_C y\,dx + x\,dy + xyz\,dz$ is not independent of path.

31–34 Determine whether or not the given set is (a) open, (b) connected, and (c) simply-connected.

31. $\{(x, y) \mid 0 < y < 3\}$ $\qquad$ **32.** $\{(x, y) \mid 1 < |x| < 2\}$

33. $\{(x, y) \mid 1 \leqslant x^2 + y^2 \leqslant 4,\ y \geqslant 0\}$

34. $\{(x, y) \mid (x, y) \neq (2, 3)\}$

35. Let $\mathbf{F}(x, y) = \dfrac{-y\,\mathbf{i} + x\,\mathbf{j}}{x^2 + y^2}$.

(a) Show that $\partial P/\partial y = \partial Q/\partial x$.

(b) Show that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is not independent of path. [*Hint:* Compute $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, where C_1 and C_2 are the upper and lower halves of the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(-1, 0)$.] Does this contradict Theorem 6?

36. (a) Suppose that $\mathbf{F}$ is an inverse square force field, that is,

$$\mathbf{F}(\mathbf{r}) = \frac{c\mathbf{r}}{|\mathbf{r}|^3}$$

for some constant c, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Find the work done by $\mathbf{F}$ in moving an object from a point P_1 along a path to a point P_2 in terms of the distances d_1 and d_2 from these points to the origin.

(b) An example of an inverse square field is the gravitational field $\mathbf{F} = -(mMG)\mathbf{r}/|\mathbf{r}|^3$ discussed in Example 4 in Section 16.1. Use part (a) to find the work done by the gravitational field when the earth moves from aphelion (at a maximum distance of 1.52×10^8 km from the sun) to perihelion (at a minimum distance of 1.47×10^8 km). (Use the values $m = 5.97 \times 10^{24}$ kg, $M = 1.99 \times 10^{30}$ kg, and $G = 6.67 \times 10^{-11}$ N·m²/kg².)

(c) Another example of an inverse square field is the electric force field $\mathbf{F} = \varepsilon qQ\mathbf{r}/|\mathbf{r}|^3$ discussed in Example 5 in Section 16.1. Suppose that an electron with a charge of -1.6×10^{-19} C is located at the origin. A positive unit charge is positioned a distance 10^{-12} m from the electron and moves to a position half that distance from the electron. Use part (a) to find the work done by the electric force field. (Use the value $\varepsilon = 8.985 \times 10^9$.)

49. Let $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$. Then

$$\begin{aligned}
\int_C \mathbf{v}\cdot d\mathbf{r} &= \int_a^b \langle v_1, v_2, v_3\rangle \cdot \langle x'(t), y'(t), z'(t)\rangle\, dt = \int_a^b \left[v_1\, x'(t) + v_2\, y'(t) + v_3\, z'(t)\right] dt \\
&= \left[v_1\, x(t) + v_2\, y(t) + v_3\, z(t)\right]_a^b = \left[v_1\, x(b) + v_2\, y(b) + v_3\, z(b)\right] - \left[v_1\, x(a) + v_2\, y(a) + v_3\, z(a)\right] \\
&= v_1\left[x(b) - x(a)\right] + v_2\left[y(b) - y(a)\right] + v_3\left[z(b) - z(a)\right] \\
&= \langle v_1, v_2, v_3\rangle \cdot \langle x(b) - x(a), y(b) - y(a), z(b) - z(a)\rangle \\
&= \langle v_1, v_2, v_3\rangle \cdot \left[\langle x(b), y(b), z(b)\rangle - \langle x(a), y(a), z(a)\rangle\right] = \mathbf{v}\cdot\left[\mathbf{r}(b) - \mathbf{r}(a)\right]
\end{aligned}$$

50. If $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ then

$$\begin{aligned}
\int_C \mathbf{r}\cdot d\mathbf{r} &= \int_a^b \langle x(t), y(t), z(t)\rangle \cdot \langle x'(t), y'(t), z'(t)\rangle\, dt = \int_a^b \left[x(t)\, x'(t) + y(t)\, y'(t) + z(t)\, z'(t)\right] dt \\
&= \left[\tfrac{1}{2}[x(t)]^2 + \tfrac{1}{2}[y(t)]^2 + \tfrac{1}{2}[z(t)]^2\right]_a^b \\
&= \tfrac{1}{2}\left\{\left([x(b)]^2 + [y(b)]^2 + [z(b)]^2\right) - \left([x(a)]^2 + [y(a)]^2 + [z(a)]^2\right)\right\} \\
&= \tfrac{1}{2}\left[|\mathbf{r}(b)|^2 - |\mathbf{r}(a)|^2\right]
\end{aligned}$$

51. The work done in moving the object is $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\, ds$. We can approximate this integral by dividing C into 7 segments of equal length $\Delta s = 2$ and approximating $\mathbf{F}\cdot\mathbf{T}$, that is, the tangential component of force, at a point (x_i^*, y_i^*) on each segment. Since C is composed of straight line segments, $\mathbf{F}\cdot\mathbf{T}$ is the scalar projection of each force vector onto C. If we choose (x_i^*, y_i^*) to be the point on the segment closest to the origin, then the work done is $\int_C \mathbf{F}\cdot\mathbf{T}\, ds \approx \sum\limits_{i=1}^{7} \left[\mathbf{F}(x_i^*, y_i^*)\cdot\mathbf{T}(x_i^*, y_i^*)\right]\Delta s = [2 + 2 + 2 + 2 + 1 + 1 + 1](2) = 22$. Thus, we estimate the work done to be approximately 22 J.

52. Use the orientation pictured in the figure. Then since $\mathbf{B}$ is tangent to any circle that lies in the plane perpendicular to the wire, $\mathbf{B} = |\mathbf{B}|\,\mathbf{T}$ where $\mathbf{T}$ is the unit tangent to the circle C: $x = r\cos\theta$, $y = r\sin\theta$. Thus $\mathbf{B} = |\mathbf{B}|\,\langle -\sin\theta, \cos\theta\rangle$. Then $\int_C \mathbf{B}\cdot d\mathbf{r} = \int_0^{2\pi} |\mathbf{B}|\,\langle -\sin\theta, \cos\theta\rangle \cdot \langle -r\sin\theta, r\cos\theta\rangle\, d\theta = \int_0^{2\pi} |\mathbf{B}|\, r\, d\theta = 2\pi r\,|\mathbf{B}|$. (Note that $|\mathbf{B}|$ here is the magnitude of the field at a distance r from the wire's center.) But by Ampere's Law $\int_C \mathbf{B}\cdot d\mathbf{r} = \mu_0 I$. Hence $|\mathbf{B}| = \mu_0 I/(2\pi r)$.

16.3 The Fundamental Theorem for Line Integrals

1. C appears to be a smooth curve, and since ∇f is continuous, we know f is differentiable. Then Theorem 2 says that the value of $\int_C \nabla f\cdot d\mathbf{r}$ is simply the difference of the values of f at the terminal and initial points of C. From the graph, this is $50 - 10 = 40$.

2. C is represented by the vector function $\mathbf{r}(t) = (t^2 + 1)\,\mathbf{i} + (t^3 + t)\,\mathbf{j}$, $0 \le t \le 1$, so $\mathbf{r}'(t) = 2t\,\mathbf{i} + (3t^2 + 1)\,\mathbf{j}$. Since $3t^2 + 1 \neq 0$, we have $\mathbf{r}'(t) \neq \mathbf{0}$, thus C is a smooth curve. ∇f is continuous, and hence f is differentiable, so by Theorem 2 we have $\int_C \nabla f\cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(2, 2) - f(1, 0) = 9 - 3 = 6$.

3. $\partial(2x - 3y)/\partial y = -3 = \partial(-3x + 4y - 8)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$ which is open and simply-connected, so by Theorem 6 $\mathbf{F}$ is conservative. Thus, there exists a function f such that $\nabla f = \mathbf{F}$, that is, $f_x(x,y) = 2x - 3y$ and $f_y(x,y) = -3x + 4y - 8$. But $f_x(x,y) = 2x - 3y$ implies $f(x,y) = x^2 - 3xy + g(y)$ and differentiating both sides of this equation with respect to y gives $f_y(x,y) = -3x + g'(y)$. Thus $-3x + 4y - 8 = -3x + g'(y)$ so $g'(y) = 4y - 8$ and $g(y) = 2y^2 - 8y + K$ where K is a constant. Hence $f(x,y) = x^2 - 3xy + 2y^2 - 8y + K$ is a potential function for $\mathbf{F}$.

4. $\partial(e^x \sin y)/\partial y = e^x \cos y = \partial(e^x \cos y)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$. Hence $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = e^x \sin y$ implies $f(x,y) = e^x \sin y + g(y)$ and $f_y(x,y) = e^x \cos y + g'(y)$. But $f_y(x,y) = e^x \cos y$ so $g'(y) = 0 \Rightarrow g(y) = K$. Then $f(x,y) = e^x \sin y + K$ is a potential function for $\mathbf{F}$.

5. $\partial(e^x \cos y)/\partial y = -e^x \sin y$, $\partial(e^x \sin y)/\partial x = e^x \sin y$. Since these are not equal, $\mathbf{F}$ is not conservative.

6. $\partial(3x^2 - 2y^2)/\partial y = -4y$, $\partial(4xy + 3)/\partial x = 4y$. Since these are not equal, $\mathbf{F}$ is not conservative.

7. $\partial(ye^x + \sin y)/\partial y = e^x + \cos y = \partial(e^x + x\cos y)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$. Hence $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = ye^x + \sin y$ implies $f(x,y) = ye^x + x\sin y + g(y)$ and $f_y(x,y) = e^x + x\cos y + g'(y)$. But $f_y(x,y) = e^x + x\cos y$ so $g(y) = K$ and $f(x,y) = ye^x + x\sin y + K$ is a potential function for $\mathbf{F}$.

8. $\partial(2xy + y^{-2})/\partial y = 2x - 2y^{-3} = \partial(x^2 - 2xy^{-3})/\partial x$ and the domain of $\mathbf{F}$ is $\{(x,y) \mid y > 0\}$ which is open and simply-connected. Hence $\mathbf{F}$ is conservative, so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = 2xy + y^{-2}$ implies $f(x,y) = x^2y + xy^{-2} + g(y)$ and $f_y(x,y) = x^2 - 2xy^{-3} + g'(y)$. But $f_y(x,y) = x^2 - 2xy^{-3}$ so $g'(y) = 0 \Rightarrow g(y) = K$. Then $f(x,y) = x^2y + xy^{-2} + K$ is a potential function for $\mathbf{F}$.

9. $\partial(\ln y + 2xy^3)/\partial y = 1/y + 6xy^2 = \partial(3x^2y^2 + x/y)/\partial x$ and the domain of $\mathbf{F}$ is $\{(x,y) \mid y > 0\}$ which is open and simply connected. Hence $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = \ln y + 2xy^3$ implies $f(x,y) = x\ln y + x^2y^3 + g(y)$ and $f_y(x,y) = x/y + 3x^2y^2 + g'(y)$. But $f_y(x,y) = 3x^2y^2 + x/y$ so $g'(y) = 0 \Rightarrow g(y) = K$ and $f(x,y) = x\ln y + x^2y^3 + K$ is a potential function for $\mathbf{F}$.

10. $$\frac{\partial(xy\cosh xy + \sinh xy)}{\partial y} = x^2y\sinh xy + x\cosh xy + x\cosh xy = x^2y\sinh xy + 2x\cosh xy = \frac{\partial(x^2\cosh xy)}{\partial x}$$
and the domain of $\mathbf{F}$ is $\mathbb{R}^2$. Thus $\mathbf{F}$ is conservative, so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = xy\cosh xy + \sinh xy$ implies $f(x,y) = x\sinh xy + g(y) \Rightarrow f_y(x,y) = x^2\cosh xy + g'(y)$. But $f_y(x,y) = x^2\cosh xy$ so $g(y) = K$ and $f(x,y) = x\sinh xy + K$ is a potential function for $\mathbf{F}$.

11. (a) $\mathbf{F}$ has continuous first-order partial derivatives and $\dfrac{\partial}{\partial y}\,2xy = 2x = \dfrac{\partial}{\partial x}\,(x^2)$ on $\mathbb{R}^2$, which is open and simply-connected. Thus, $\mathbf{F}$ is conservative by Theorem 6. Then we know that the line integral of $\mathbf{F}$ is independent of path; in particular, the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends only on the endpoints of C. Since all three curves have the same initial and terminal points, $\int_C \mathbf{F} \cdot d\mathbf{r}$ will have the same value for each curve.

(b) We first find a potential function f, so that $\nabla f = \mathbf{F}$. We know $f_x(x,y) = 2xy$ and $f_y(x,y) = x^2$. Integrating $f_x(x,y)$ with respect to x, we have $f(x,y) = x^2y + g(y)$. Differentiating both sides with respect to y gives $f_y(x,y) = x^2 + g'(y)$, so we must have $x^2 + g'(y) = x^2 \quad\Rightarrow\quad g'(y) = 0 \quad\Rightarrow\quad g(y) = K$, a constant. Thus $f(x,y) = x^2y + K$. All three curves start at $(1,2)$ and end at $(3,2)$, so by Theorem 2, $\int_C \mathbf{F}\cdot d\mathbf{r} = f(3,2) - f(1,2) = 18 - 2 = 16$ for each curve.

12. (a) $f_x(x,y) = x^2$ implies $f(x,y) = \frac{1}{3}x^3 + g(y)$ and $f_y(x,y) = 0 + g'(y)$. But $f_y(x,y) = y^2$ so $g'(y) = y^2 \quad\Rightarrow\quad g(y) = \frac{1}{3}y^3 + K$. We can take $K = 0$, so $f(x,y) = \frac{1}{3}x^3 + \frac{1}{3}y^3$.

(b) $\int_C \mathbf{F}\cdot d\mathbf{r} = f(2,8) - f(-1,2) = \left(\frac{8}{3} + \frac{512}{3}\right) - \left(-\frac{1}{3} + \frac{8}{3}\right) = 171$.

13. (a) $f_x(x,y) = xy^2$ implies $f(x,y) = \frac{1}{2}x^2y^2 + g(y)$ and $f_y(x,y) = x^2y + g'(y)$. But $f_y(x,y) = x^2y$ so $g'(y) = 0 \quad\Rightarrow\quad g(y) = K$, a constant. We can take $K = 0$, so $f(x,y) = \frac{1}{2}x^2y^2$.

(b) The initial point of C is $\mathbf{r}(0) = (0,1)$ and the terminal point is $\mathbf{r}(1) = (2,1)$, so $\int_C \mathbf{F}\cdot d\mathbf{r} = f(2,1) - f(0,1) = 2 - 0 = 2$.

14. (a) $f_y(x,y) = x^2e^{xy}$ implies $f(x,y) = xe^{xy} + g(x) \quad\Rightarrow\quad f_x(x,y) = xye^{xy} + e^{xy} + g'(x) = (1+xy)e^{xy} + g'(x)$. But $f_x(x,y) = (1+xy)e^{xy}$ so $g'(x) = 0 \quad\Rightarrow\quad g(x) = K$. We can take $K = 0$, so $f(x,y) = xe^{xy}$.

(b) The initial point of C is $\mathbf{r}(0) = (1,0)$ and the terminal point is $\mathbf{r}(\pi/2) = (0,2)$, so $\int_C \mathbf{F}\cdot d\mathbf{r} = f(0,2) - f(1,0) = 0 - e^0 = -1$.

15. (a) $f_x(x,y,z) = yz$ implies $f(x,y,z) = xyz + g(y,z)$ and so $f_y(x,y,z) = xz + g_y(y,z)$. But $f_y(x,y,z) = xz$ so $g_y(y,z) = 0 \quad\Rightarrow\quad g(y,z) = h(z)$. Thus $f(x,y,z) = xyz + h(z)$ and $f_z(x,y,z) = xy + h'(z)$. But $f_z(x,y,z) = xy + 2z$, so $h'(z) = 2z \quad\Rightarrow\quad h(z) = z^2 + K$. Hence $f(x,y,z) = xyz + z^2$ (taking $K = 0$).

(b) $\int_C \mathbf{F}\cdot d\mathbf{r} = f(4,6,3) - f(1,0,-2) = 81 - 4 = 77$.

16. (a) $f_x(x,y,z) = y^2z + 2xz^2$ implies $f(x,y,z) = xy^2z + x^2z^2 + g(y,z)$ and so $f_y(x,y,z) = 2xyz + g_y(y,z)$. But $f_y(x,y,z) = 2xyz$ so $g_y(y,z) = 0 \quad\Rightarrow\quad g(y,z) = h(z)$. Thus $f(x,y,z) = xy^2z + x^2z^2 + h(z)$ and $f_z(x,y,z) = xy^2 + 2x^2z + h'(z)$. But $f_z(x,y,z) = xy^2 + 2x^2z$, so $h'(z) = 0 \quad\Rightarrow\quad h(z) = K$. Hence $f(x,y,z) = xy^2z + x^2z^2$ (taking $K = 0$).

(b) $t = 0$ corresponds to the point $(0,1,0)$ and $t = 1$ corresponds to $(1,2,1)$, so $\int_C \mathbf{F}\cdot d\mathbf{r} = f(1,2,1) - f(0,1,0) = 5 - 0 = 5$.

17. (a) $f_x(x,y,z) = yze^{xz}$ implies $f(x,y,z) = ye^{xz} + g(y,z)$ and so $f_y(x,y,z) = e^{xz} + g_y(y,z)$. But $f_y(x,y,z) = e^{xz}$ so $g_y(y,z) = 0 \quad\Rightarrow\quad g(y,z) = h(z)$. Thus $f(x,y,z) = ye^{xz} + h(z)$ and $f_z(x,y,z) = xye^{xz} + h'(z)$. But $f_z(x,y,z) = xye^{xz}$, so $h'(z) = 0 \quad\Rightarrow\quad h(z) = K$. Hence $f(x,y,z) = ye^{xz}$ (taking $K = 0$).

(b) $\mathbf{r}(0) = \langle 1,-1,0\rangle$, $\mathbf{r}(2) = \langle 5,3,0\rangle$ so $\int_C \mathbf{F}\cdot d\mathbf{r} = f(5,3,0) - f(1,-1,0) = 3e^0 + e^0 = 4$.

18. (a) $f_x(x,y,z) = \sin y$ implies $f(x,y,z) = x\sin y + g(y,z)$ and so $f_y(x,y,z) = x\cos y + g_y(y,z)$. But $f_y(x,y,z) = x\cos y + \cos z$ so $g_y(y,z) = \cos z \;\Rightarrow\; g(y,z) = y\cos z + h(z)$. Thus $f(x,y,z) = x\sin y + y\cos z + h(z)$ and $f_z(x,y,z) = -y\sin z + h'(z)$. But $f_z(x,y,z) = -y\sin z$, so $h'(z) = 0 \;\Rightarrow$ $h(z) = K$. Hence $f(x,y,z) = x\sin y + y\cos z$ (taking $K = 0$).

(b) $\mathbf{r}(0) = \langle 0,0,0\rangle$, $\mathbf{r}(\pi/2) = \langle 1,\pi/2,\pi\rangle$ so $\int_C \mathbf{F}\cdot d\mathbf{r} = f(1,\pi/2,\pi) - f(0,0,0) = 1 - \frac{\pi}{2} - 0 = 1 - \frac{\pi}{2}$.

19. The functions $2xe^{-y}$ and $2y - x^2e^{-y}$ have continuous first-order derivatives on $\mathbb{R}^2$ and $\dfrac{\partial}{\partial y}\left(2xe^{-y}\right) = -2xe^{-y} = \dfrac{\partial}{\partial x}\left(2y - x^2e^{-y}\right)$, so $\mathbf{F}(x,y) = 2xe^{-y}\,\mathbf{i} + \left(2y - x^2e^{-y}\right)\mathbf{j}$ is a conservative vector field by Theorem 6 and hence the line integral is independent of path. Thus a potential function f exists, and $f_x(x,y) = 2xe^{-y}$ implies $f(x,y) = x^2e^{-y} + g(y)$ and $f_y(x,y) = -x^2e^{-y} + g'(y)$. But $f_y(x,y) = 2y - x^2e^{-y}$ so $g'(y) = 2y \;\Rightarrow\; g(y) = y^2 + K$. We can take $K = 0$, so $f(x,y) = x^2e^{-y} + y^2$. Then $\int_C 2xe^{-y}\,dx + (2y - x^2e^{-y})\,dy = f(2,1) - f(1,0) = 4e^{-1} + 1 - 1 = 4/e$.

20. The functions $\sin y$ and $x\cos y - \sin y$ have continuous first-order derivatives on $\mathbb{R}^2$ and $\dfrac{\partial}{\partial y}(\sin y) = \cos y = \dfrac{\partial}{\partial x}(x\cos y - \sin y)$, so $\mathbf{F}(x,y) = \sin y\,\mathbf{i} + (x\cos y - \sin y)\,\mathbf{j}$ is a conservative vector field by Theorem 6 and hence the line integral is independent of path. Thus a potential function f exists, and $f_x(x,y) = \sin y$ implies $f(x,y) = x\sin y + g(y)$ and $f_y(x,y) = x\cos y + g'(y)$. But $f_y(x,y) = x\cos y - \sin y$ so $g'(y) = -\sin y \;\Rightarrow\; g(y) = \cos y + K$. We can take $K = 0$, so $f(x,y) = x\sin y + \cos y$. Then $\int_C \sin y\,dx + (x\cos y - \sin y)\,dy = f(1,\pi) - f(2,0) = -1 - 1 = -2$.

21. If $\mathbf{F}$ is conservative, then $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path. This means that the work done along all piecewise-smooth curves that have the described initial and terminal points is the same. Your reply: It doesn't matter which curve is chosen.

22. The curves C_1 and C_2 connect the same two points but $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$. Thus $\mathbf{F}$ is not independent of path, and therefore is not conservative.

23. $\mathbf{F}(x,y) = 2y^{3/2}\,\mathbf{i} + 3x\sqrt{y}\,\mathbf{j}$, $W = \int_C \mathbf{F}\cdot d\mathbf{r}$. Since $\partial(2y^{3/2})/\partial y = 3\sqrt{y} = \partial(3x\sqrt{y}\,)/\partial x$, there exists a function f such that $\nabla f = \mathbf{F}$. In fact, $f_x(x,y) = 2y^{3/2} \;\Rightarrow\; f(x,y) = 2xy^{3/2} + g(y) \;\Rightarrow\; f_y(x,y) = 3xy^{1/2} + g'(y)$. But $f_y(x,y) = 3x\sqrt{y}$ so $g'(y) = 0$ or $g(y) = K$. We can take $K = 0 \;\Rightarrow\; f(x,y) = 2xy^{3/2}$. Thus $W = \int_C \mathbf{F}\cdot d\mathbf{r} = f(2,4) - f(1,1) = 2(2)(8) - 2(1) = 30$.

24. $\mathbf{F}(x,y) = e^{-y}\,\mathbf{i} - xe^{-y}\,\mathbf{j}$, $W = \int_C \mathbf{F}\cdot d\mathbf{r}$. Since $\dfrac{\partial}{\partial y}\left(e^{-y}\right) = -e^{-y} = \dfrac{\partial}{\partial x}\left(-xe^{-y}\right)$, there exists a function f such that $\nabla f = \mathbf{F}$. In fact, $f_x = e^{-y} \;\Rightarrow\; f(x,y) = xe^{-y} + g(y) \;\Rightarrow\; f_y = -xe^{-y} + g'(y) \;\Rightarrow\; g'(y) = 0$, so we can take $f(x,y) = xe^{-y}$ as a potential function for $\mathbf{F}$. Thus $W = \int_C \mathbf{F}\cdot d\mathbf{r} = f(2,0) - f(0,1) = 2 - 0 = 2$.

0.0.9 Questions Solutions Lagrange Max Min on Boundary

For functions of two variables the method of Lagrange multipliers is similar to the method just described. To find the extreme values of $f(x, y)$ subject to the constraint $g(x, y) = k$, we look for values of x, y, and λ such that

$$\nabla f(x, y) = \lambda \nabla g(x, y) \qquad \text{and} \qquad g(x, y) = k$$

This amounts to solving three equations in three unknowns:

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad g(x, y) = k$$

Our first illustration of Lagrange's method is to reconsider the problem given in Example 6 in Section 14.7.

V EXAMPLE 1 A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

SOLUTION As in Example 6 in Section 14.7, we let x, y, and z be the length, width, and height, respectively, of the box in meters. Then we wish to maximize

$$V = xyz$$

subject to the constraint

$$g(x, y, z) = 2xz + 2yz + xy = 12$$

Using the method of Lagrange multipliers, we look for values of x, y, z, and λ such that $\nabla V = \lambda \nabla g$ and $g(x, y, z) = 12$. This gives the equations

$$V_x = \lambda g_x$$

$$V_y = \lambda g_y$$

$$V_z = \lambda g_z$$

$$2xz + 2yz + xy = 12$$

which become

$$\boxed{2} \qquad yz = \lambda(2z + y)$$

$$\boxed{3} \qquad xz = \lambda(2z + x)$$

$$\boxed{4} \qquad xy = \lambda(2x + 2y)$$

$$\boxed{5} \qquad 2xz + 2yz + xy = 12$$

There are no general rules for solving systems of equations. Sometimes some ingenuity is required. In the present example you might notice that if we multiply $\boxed{2}$ by x, $\boxed{3}$ by y, and $\boxed{4}$ by z, then the left sides of these equations will be identical. Doing this, we have

Another method for solving the system of equations (2–5) is to solve each of Equations 2, 3, and 4 for λ and then to equate the resulting expressions.

$$\boxed{6} \qquad xyz = \lambda(2xz + xy)$$

$$\boxed{7} \qquad xyz = \lambda(2yz + xy)$$

$$\boxed{8} \qquad xyz = \lambda(2xz + 2yz)$$

We observe that $\lambda \neq 0$ because $\lambda = 0$ would imply $yz = xz = xy = 0$ from $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ and this would contradict $\boxed{5}$. Therefore, from $\boxed{6}$ and $\boxed{7}$, we have

$$2xz + xy = 2yz + xy$$

which gives $xz = yz$. But $z \neq 0$ (since $z = 0$ would give $V = 0$), so $x = y$. From $\boxed{7}$ and $\boxed{8}$ we have

$$2yz + xy = 2xz + 2yz$$

which gives $2xz = xy$ and so (since $x \neq 0$) $y = 2z$. If we now put $x = y = 2z$ in $\boxed{5}$, we get

$$4z^2 + 4z^2 + 4z^2 = 12$$

Since x, y, and z are all positive, we therefore have $z = 1$ and so $x = 2$ and $y = 2$. This agrees with our answer in Section 14.7.

In geometric terms, Example 2 asks for the highest and lowest points on the curve C in Figure 2 that lie on the paraboloid $z = x^2 + 2y^2$ and directly above the constraint circle $x^2 + y^2 = 1$.

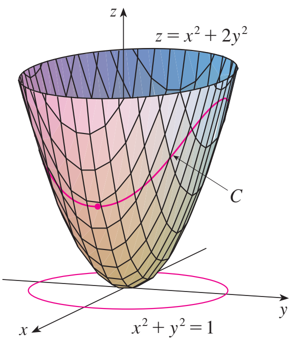


FIGURE 2

V EXAMPLE 2 Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

SOLUTION We are asked for the extreme values of f subject to the constraint $g(x, y) = x^2 + y^2 = 1$. Using Lagrange multipliers, we solve the equations $\nabla f = \lambda \nabla g$ and $g(x, y) = 1$, which can be written as

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad g(x, y) = 1$$

or as

$$2x = 2x\lambda \tag{9}$$

$$4y = 2y\lambda \tag{10}$$

$$x^2 + y^2 = 1 \tag{11}$$

From $\boxed{9}$ we have $x = 0$ or $\lambda = 1$. If $x = 0$, then $\boxed{11}$ gives $y = \pm 1$. If $\lambda = 1$, then $y = 0$ from $\boxed{10}$, so then $\boxed{11}$ gives $x = \pm 1$. Therefore f has possible extreme values at the points $(0, 1)$, $(0, -1)$, $(1, 0)$, and $(-1, 0)$. Evaluating f at these four points, we find that

$$f(0, 1) = 2 \qquad f(0, -1) = 2 \qquad f(1, 0) = 1 \qquad f(-1, 0) = 1$$

Therefore the maximum value of f on the circle $x^2 + y^2 = 1$ is $f(0, \pm 1) = 2$ and the minimum value is $f(\pm 1, 0) = 1$. Checking with Figure 2, we see that these values look reasonable.

The geometry behind the use of Lagrange multipliers in Example 2 is shown in Figure 3. The extreme values of $f(x, y) = x^2 + 2y^2$ correspond to the level curves that touch the circle $x^2 + y^2 = 1$.

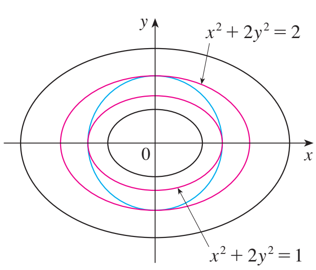


FIGURE 3

EXAMPLE 3 Find the extreme values of $f(x, y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \leqslant 1$.

SOLUTION According to the procedure in (14.7.9), we compare the values of f at the critical points with values at the points on the boundary. Since $f_x = 2x$ and $f_y = 4y$, the only critical point is $(0, 0)$. We compare the value of f at that point with the extreme values on the boundary from Example 2:

$$f(0, 0) = 0 \qquad f(\pm 1, 0) = 1 \qquad f(0, \pm 1) = 2$$

Therefore the maximum value of f on the disk $x^2 + y^2 \leqslant 1$ is $f(0, \pm 1) = 2$ and the minimum value is $f(0, 0) = 0$.

EXAMPLE 4 Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$.

SOLUTION The distance from a point (x, y, z) to the point $(3, 1, -1)$ is

$$d = \sqrt{(x - 3)^2 + (y - 1)^2 + (z + 1)^2}$$

but the algebra is simpler if we instead maximize and minimize the square of the distance:

$$d^2 = f(x, y, z) = (x - 3)^2 + (y - 1)^2 + (z + 1)^2$$

The constraint is that the point (x, y, z) lies on the sphere, that is,

$$g(x, y, z) = x^2 + y^2 + z^2 = 4$$

According to the method of Lagrange multipliers, we solve $\nabla f = \lambda \nabla g$, $g = 4$. This gives

$$\boxed{12} \qquad 2(x - 3) = 2x\lambda$$

$$\boxed{13} \qquad 2(y - 1) = 2y\lambda$$

$$\boxed{14} \qquad 2(z + 1) = 2z\lambda$$

$$\boxed{15} \qquad x^2 + y^2 + z^2 = 4$$

The simplest way to solve these equations is to solve for x, y, and z in terms of λ from $\boxed{12}$, $\boxed{13}$, and $\boxed{14}$, and then substitute these values into $\boxed{15}$. From $\boxed{12}$ we have

$$x - 3 = x\lambda \qquad \text{or} \qquad x(1 - \lambda) = 3 \qquad \text{or} \qquad x = \frac{3}{1 - \lambda}$$

Figure 4 shows the sphere and the nearest point P in Example 4. Can you see how to find the coordinates of P without using calculus?

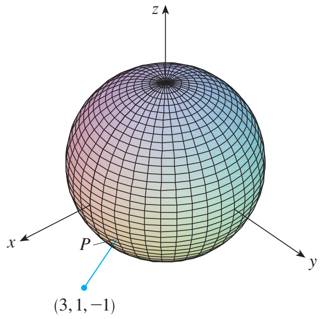


FIGURE 4

[Note that $1 - \lambda \neq 0$ because $\lambda = 1$ is impossible from $\boxed{12}$.] Similarly, $\boxed{13}$ and $\boxed{14}$ give

$$y = \frac{1}{1 - \lambda} \qquad z = -\frac{1}{1 - \lambda}$$

Therefore, from $\boxed{15}$, we have

$$\frac{3^2}{(1 - \lambda)^2} + \frac{1^2}{(1 - \lambda)^2} + \frac{(-1)^2}{(1 - \lambda)^2} = 4$$

which gives $(1 - \lambda)^2 = \frac{11}{4}$, $1 - \lambda = \pm\sqrt{11}/2$, so

$$\lambda = 1 \pm \frac{\sqrt{11}}{2}$$

These values of λ then give the corresponding points (x, y, z):

$$\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right) \qquad \text{and} \qquad \left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$$

It's easy to see that f has a smaller value at the first of these points, so the closest point is $\left(6/\sqrt{11}, 2/\sqrt{11}, -2/\sqrt{11}\right)$ and the farthest is $\left(-6/\sqrt{11}, -2/\sqrt{11}, 2/\sqrt{11}\right)$.

Two Constraints

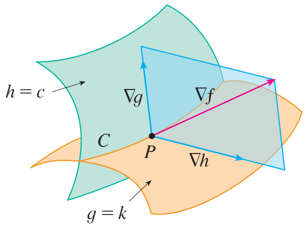


FIGURE 5

Suppose now that we want to find the maximum and minimum values of a function $f(x, y, z)$ subject to two constraints (side conditions) of the form $g(x, y, z) = k$ and $h(x, y, z) = c$. Geometrically, this means that we are looking for the extreme values of f when (x, y, z) is restricted to lie on the curve of intersection C of the level surfaces $g(x, y, z) = k$ and $h(x, y, z) = c$. (See Figure 5.) Suppose f has such an extreme value at a point $P(x_0, y_0, z_0)$.

0.0.10 Questions Solutions Limits

is a rational function of three variables and so is continuous at every point in $\mathbb{R}^3$ except where $x^2 + y^2 + z^2 = 1$. In other words, it is discontinuous on the sphere with center the origin and radius 1.

If we use the vector notation introduced at the end of Section 14.1, then we can write the definitions of a limit for functions of two or three variables in a single compact form as follows.

> $\boxed{5}$ If f is defined on a subset D of $\mathbb{R}^n$, then $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ means that for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that
>
> $$\text{if} \quad \mathbf{x} \in D \quad \text{and} \quad 0 < |\mathbf{x} - \mathbf{a}| < \delta \quad \text{then} \quad |f(\mathbf{x}) - L| < \varepsilon$$

Notice that if $n = 1$, then $\mathbf{x} = x$ and $\mathbf{a} = a$, and $\boxed{5}$ is just the definition of a limit for functions of a single variable. For the case $n = 2$, we have $\mathbf{x} = \langle x, y\rangle$, $\mathbf{a} = \langle a, b\rangle$, and $|\mathbf{x} - \mathbf{a}| = \sqrt{(x-a)^2 + (y-b)^2}$, so $\boxed{5}$ becomes Definition 1. If $n = 3$, then $\mathbf{x} = \langle x, y, z\rangle$, $\mathbf{a} = \langle a, b, c\rangle$, and $\boxed{5}$ becomes the definition of a limit of a function of three variables. In each case the definition of continuity can be written as

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$$

14.2 Exercises

1. Suppose that $\lim_{(x,y)\to(3,1)} f(x, y) = 6$. What can you say about the value of $f(3, 1)$? What if f is continuous?

2. Explain why each function is continuous or discontinuous.
(a) The outdoor temperature as a function of longitude, latitude, and time
(b) Elevation (height above sea level) as a function of longitude, latitude, and time
(c) The cost of a taxi ride as a function of distance traveled and time

3–4 Use a table of numerical values of $f(x, y)$ for (x, y) near the origin to make a conjecture about the value of the limit of $f(x, y)$ as $(x, y) \to (0, 0)$. Then explain why your guess is correct.

3. $f(x, y) = \dfrac{x^2y^3 + x^3y^2 - 5}{2 - xy}$ **4.** $f(x, y) = \dfrac{2xy}{x^2 + 2y^2}$

5–22 Find the limit, if it exists, or show that the limit does not exist.

5 **5.** $\displaystyle\lim_{(x,y)\to(1,2)} (5x^3 - x^2y^2)$ **6.** $\displaystyle\lim_{(x,y)\to(1,-1)} e^{-xy}\cos(x + y)$

7 **7.** $\displaystyle\lim_{(x,y)\to(2,1)} \frac{4 - xy}{x^2 + 3y^2}$ **8.** $\displaystyle\lim_{(x,y)\to(1,0)} \ln\left(\frac{1 + y^2}{x^2 + xy}\right)$

9 **9.** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 - 4y^2}{x^2 + 2y^2}$ **10.** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{5y^4\cos^2 x}{x^4 + y^4}$

11 **11.** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2\sin^2 x}{x^4 + y^4}$ **12.** $\displaystyle\lim_{(x,y)\to(1,0)} \frac{xy - y}{(x-1)^2 + y^2}$

13. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2 + y^2}}$ **14.** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2 + y^2}$ 14

15 **15.** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2ye^y}{x^4 + 4y^2}$ **16.** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2\sin^2 y}{x^2 + 2y^2}$

17 **17.** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$ **18.** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2 + y^8}$

19. $\displaystyle\lim_{(x,y,z)\to(\pi,0,1/3)} e^{y^2}\tan(xz)$

20. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz}{x^2 + y^2 + z^2}$

21. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz^2 + xz^2}{x^2 + y^2 + z^4}$

22. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{yz}{x^2 + 4y^2 + 9z^2}$

23–24 Use a computer graph of the function to explain why the limit does not exist.

23. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2 + 3xy + 4y^2}{3x^2 + 5y^2}$ **24.** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2 + y^6}$

Graphing calculator or computer required **1.** Homework Hints available at stewartcalculus.com

4. We make a table of values of $f(x,y) = \dfrac{2xy}{x^2+2y^2}$ for a set of (x,y) points near the origin.

$x \backslash y$	-0.3	-0.2	-0.1	0	0.1	0.2	0.3
-0.3	0.667	0.706	0.545	0.000	-0.545	-0.706	-0.667
-0.2	0.545	0.667	0.667	0.000	-0.667	-0.667	-0.545
-0.1	0.316	0.444	0.667	0.000	-0.667	-0.444	-0.316
0	0.000	0.000	0.000		0.000	0.000	0.000
0.1	-0.316	-0.444	-0.667	0.000	0.667	0.444	0.316
0.2	-0.545	-0.667	-0.667	0.000	0.667	0.667	0.545
0.3	-0.667	-0.706	-0.545	0.000	0.545	0.706	0.667

It appears from the table that the values of $f(x,y)$ are not approaching a single value as (x,y) approaches the origin. For verification, if we first approach $(0,0)$ along the x-axis, we have $f(x,0)=0$, so $f(x,y)\to 0$. But if we approach $(0,0)$ along the line $y=x$, $f(x,x) = \dfrac{2x^2}{x^2+2x^2} = \dfrac{2}{3}$ $(x \neq 0)$, so $f(x,y) \to \frac{2}{3}$. Since f approaches different values along different paths to the origin, this limit does not exist.

5. $f(x,y) = 5x^3 - x^2y^2$ is a polynomial, and hence continuous, so $\displaystyle\lim_{(x,y)\to(1,2)} f(x,y) = f(1,2) = 5(1)^3 - (1)^2(2)^2 = 1$.

6. $-xy$ is a polynomial and therefore continuous. Since e^t is a continuous function, the composition e^{-xy} is also continuous. Similarly, $x+y$ is a polynomial and $\cos t$ is a continuous function, so the composition $\cos(x+y)$ is continuous. The product of continuous functions is continuous, so $f(x,y) = e^{-xy}\cos(x+y)$ is a continuous function and $\displaystyle\lim_{(x,y)\to(1,-1)} f(x,y) = f(1,-1) = e^{-(1)(-1)}\cos(1+(-1)) = e^1\cos 0 = e$.

7. $f(x,y) = \dfrac{4-xy}{x^2+3y^2}$ is a rational function and hence continuous on its domain.

$(2,1)$ is in the domain of f, so f is continuous there and $\displaystyle\lim_{(x,y)\to(2,1)} f(x,y) = f(2,1) = \frac{4-(2)(1)}{(2)^2+3(1)^2} = \frac{2}{7}$.

8. $\dfrac{1+y^2}{x^2+xy}$ is a rational function and hence continuous on its domain, which includes $(1,0)$. $\ln t$ is a continuous function for $t>0$, so the composition $f(x,y) = \ln\left(\dfrac{1+y^2}{x^2+xy}\right)$ is continuous wherever $\dfrac{1+y^2}{x^2+xy} > 0$. In particular, f is continuous at $(1,0)$ and so $\displaystyle\lim_{(x,y)\to(1,0)} f(x,y) = f(1,0) = \ln\left(\frac{1+0^2}{1^2+1\cdot 0}\right) = \ln\frac{1}{1} = 0$.

9. $f(x,y) = (x^4-4y^2)/(x^2+2y^2)$. First approach $(0,0)$ along the x-axis. Then $f(x,0) = x^4/x^2 = x^2$ for $x\neq 0$, so $f(x,y)\to 0$. Now approach $(0,0)$ along the y-axis. For $y\neq 0$, $f(0,y) = -4y^2/2y^2 = -2$, so $f(x,y)\to -2$. Since f has two different limits along two different lines, the limit does not exist.

10. $f(x,y) = (5y^4 \cos^2 x)/(x^4 + y^4)$. First approach $(0,0)$ along the x-axis. Then $f(x,0) = 0/x^4 = 0$ for $x \neq 0$, so $f(x,y) \to 0$. Next approach $(0,0)$ along the y-axis. For $y \neq 0$, $f(0,y) = 5y^4/y^4 = 5$, so $f(x,y) \to 5$. Since f has two different limits along two different lines, the limit does not exist.

11. $f(x,y) = (y^2 \sin^2 x)/(x^4 + y^4)$. On the x-axis, $f(x,0) = 0$ for $x \neq 0$, so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. Approaching $(0,0)$ along the line $y = x$, $f(x,x) = \dfrac{x^2 \sin^2 x}{x^4 + x^4} = \dfrac{\sin^2 x}{2x^2} = \dfrac{1}{2}\left(\dfrac{\sin x}{x}\right)^2$ for $x \neq 0$ and $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$, so $f(x,y) \to \frac{1}{2}$. Since f has two different limits along two different lines, the limit does not exist.

12. $f(x,y) = \dfrac{xy - y}{(x-1)^2 + y^2}$. On the x-axis, $f(x,0) = 0/(x-1)^2 = 0$ for $x \neq 1$, so $f(x,y) \to 0$ as $(x,y) \to (1,0)$ along the x-axis. Approaching $(1,0)$ along the line $y = x - 1$, $f(x, x-1) = \dfrac{x(x-1) - (x-1)}{(x-1)^2 + (x-1)^2} = \dfrac{(x-1)^2}{2(x-1)^2} = \dfrac{1}{2}$ for $x \neq 1$, so $f(x,y) \to \frac{1}{2}$ along this line. Thus the limit does not exist.

13. $f(x,y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$. We can see that the limit along any line through $(0,0)$ is 0, as well as along other paths through $(0,0)$ such as $x = y^2$ and $y = x^2$. So we suspect that the limit exists and equals 0; we use the Squeeze Theorem to prove our assertion. $0 \leq \left|\dfrac{xy}{\sqrt{x^2 + y^2}}\right| \leq |x|$ since $|y| \leq \sqrt{x^2 + y^2}$, and $|x| \to 0$ as $(x,y) \to (0,0)$. So $\lim\limits_{(x,y) \to (0,0)} f(x,y) = 0$.

14. $f(x,y) = \dfrac{x^4 - y^4}{x^2 + y^2} = \dfrac{(x^2 + y^2)(x^2 - y^2)}{x^2 + y^2} = x^2 - y^2$ for $(x,y) \neq (0,0)$. Thus the limit as $(x,y) \to (0,0)$ is 0.

15. Let $f(x,y) = \dfrac{x^2 y e^y}{x^4 + 4y^2}$. Then $f(x,0) = 0$ for $x \neq 0$, so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. Approaching $(0,0)$ along the y-axis or the line $y = x$ also gives a limit of 0. But $f\left(x, x^2\right) = \dfrac{x^2 x^2 e^{x^2}}{x^4 + 4(x^2)^2} = \dfrac{x^4 e^{x^2}}{5x^4} = \dfrac{e^{x^2}}{5}$ for $x \neq 0$, so $f(x,y) \to e^0/5 = \frac{1}{5}$ as $(x,y) \to (0,0)$ along the parabola $y = x^2$. Thus the limit doesn't exist.

16. We can use the Squeeze Theorem to show that $\lim\limits_{(x,y) \to (0,0)} \dfrac{x^2 \sin^2 y}{x^2 + 2y^2} = 0$:

$0 \leq \dfrac{x^2 \sin^2 y}{x^2 + 2y^2} \leq \sin^2 y$ since $\dfrac{x^2}{x^2 + 2y^2} \leq 1$, and $\sin^2 y \to 0$ as $(x,y) \to (0,0)$, so $\lim\limits_{(x,y) \to (0,0)} \dfrac{x^2 \sin^2 y}{x^2 + 2y^2} = 0$.

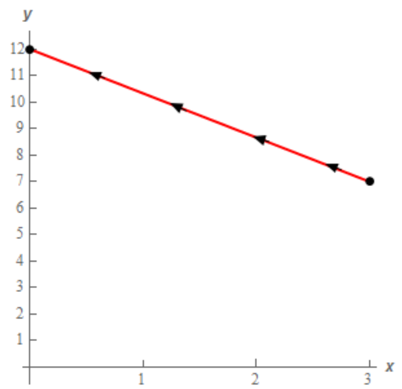
17.
$$\begin{aligned}
\lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1} &= \lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1} \cdot \frac{\sqrt{x^2 + y^2 + 1} + 1}{\sqrt{x^2 + y^2 + 1} + 1} \\
&= \lim_{(x,y) \to (0,0)} \frac{\left(x^2 + y^2\right)\left(\sqrt{x^2 + y^2 + 1} + 1\right)}{x^2 + y^2} = \lim_{(x,y) \to (0,0)} \left(\sqrt{x^2 + y^2 + 1} + 1\right) = 2
\end{aligned}$$

0.0.11 Questions Solutions Line Integral with respect dr

1. Line Integrals Of Vector Fields - Practice Problems Solutions

1. Evaluate $\displaystyle\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y) = y^2\,\vec{i} + (3x - 6y)\,\vec{j}$ and C is the line segment from $(3,7)$ to $(0,12)$.

Here is a quick sketch of C with the direction specified in the problem statement shown.



Next, we need to parameterize the curve.

$$\vec{r}\,(t) = (1-t)\,\langle 3,7\rangle + t\,\langle 0,12\rangle = \langle 3-3t, 7+5t\rangle \qquad 0 \le t \le 1$$

In order to evaluate this line integral we'll need the dot product of the vector field (evaluated at the along the curve) and the derivative of the parameterization.

Here is the vector field evaluated along the curve (*i.e.* plug in x and y from the parameterization into the vector field).

$$\vec{F}\,(\vec{r}\,(t)) = (7+5t)^2\,\vec{i} + (3\,(3-3t) - 6\,(7+5t))\,\vec{j} = (7+5t)^2\,\vec{i} + (-33-39t)\,\vec{j}$$

The derivative of the parameterization is,

$$\vec{r}\,'\,(t) = \langle -3, 5\rangle$$

Finally, the dot product of the vector field and the derivative of the parameterization.

$$\vec{F}\,(\vec{r}\,(t)) \cdot \vec{r}\,'\,(t) = -3(7+5t)^2 - 5\,(33+39t)$$

Now all we need to do is evaluate the integral.

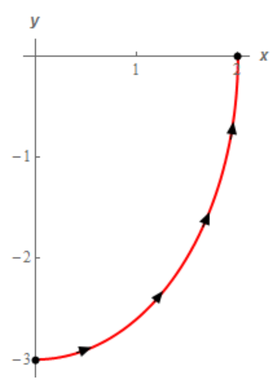
$$\int\limits_C \vec{F} \cdot d\vec{r} = \int_0^1 -3(7+5t)^2 - 5\,(33+39t)\;dt$$

$$= \left[-\frac{1}{5}(7+5t)^3 - 165t - \frac{195}{2}t^2\right]\Bigg|_0^1 = \boxed{-\frac{1079}{2}}$$

2 : Line Integrals Of Vector Fields - Practice Problems Solutions

2. Evaluate $\displaystyle\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y) = (x+y)\,\vec{i} + (1-x)\,\vec{j}$ and C is the portion of $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ that is in the 4th quadrant with the counter clockwise rotation.

Here is a quick sketch of C with the direction specified in the problem statement shown.



Next, we need to parameterize the curve.

$$\vec{r}(t) = \langle 2\cos(t), 3\sin(t)\rangle \qquad \frac{3}{2}\pi \le t \le 2\pi$$

In order to evaluate this line integral we'll need the dot product of the vector field (evaluated at the along the curve) and the derivative of the parameterization.

Here is the vector field evaluated along the curve (*i.e.* plug in x and y from the parameterization into the vector field).

$$\vec{F}(\vec{r}(t)) = (2\cos(t) + 3\sin(t))\,\vec{i} + (1 - 2\cos(t))\,\vec{j}$$

The derivative of the parameterization is,

$$\vec{r}'(t) = \langle -2\sin(t), 3\cos(t)\rangle$$

Finally, the dot product of the vector field and the derivative of the parameterization.

$$\begin{aligned}\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) &= (2\cos(t) + 3\sin(t))(-2\sin(t)) + (1 - 2\cos(t))(3\cos(t)) \\ &= -4\cos(t)\sin(t) - 6\sin^2(t) + 3\cos(t) - 6\cos^2(t) \\ &= -4\cos(t)\sin(t) - 6\left[\sin^2(t) + \cos^2(t)\right] + 3\cos(t) \\ &= -2\sin(2t) + 3\cos(t) - 6\end{aligned}$$

Make sure that you simplify the dot product with an eye towards doing the integral! In this case that meant using the double angle formula for sine to "simplify" the first term for the integral.

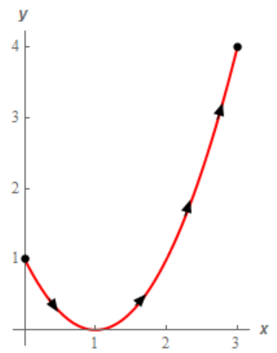
Now all we need to do is evaluate the integral.

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_{\frac{3}{2}\pi}^{2\pi} -2\sin(2t) + 3\cos(t) - 6\,dt \\ &= \left[\cos(2t) + 3\sin(t) - 6t\right]\Big|_{\frac{3}{2}\pi}^{2\pi} = \boxed{5 - 3\pi}\end{aligned}$$

3 Line Integrals Of Vector Fields - Practice Problems Solutions

3. Evaluate $\displaystyle\int_C \vec F \cdot d\vec r$ where $\vec F(x,y) = y^2\,\vec i + (x^2 - 4)\,\vec j$ and C is the portion of $y = (x-1)^2$ from $x = 0$ to $x = 3$.

Here is a quick sketch of C with the direction specified in the problem statement shown.



Next, we need to parameterize the curve.

$$\vec r(t) = \left\langle t, (t-1)^2 \right\rangle \qquad 0 \le t \le 3$$

In order to evaluate this line integral we'll need the dot product of the vector field (evaluated at the along the curve) and the derivative of the parameterization.

Here is the vector field evaluated along the curve (*i.e.* plug in x and y from the parameterization into the vector field).

$$\vec{F}\left(\vec{r}\left(t\right)\right) = \left[\left(t-1\right)^2\right]^2 \vec{i} + \left(\left(t\right)^2 - 4\right)\vec{j} = \left(t-1\right)^4 \vec{i} + \left(t^2 - 4\right)\vec{j}$$

The derivative of the parameterization is,

$$\vec{r}'\left(t\right) = \left\langle 1, 2\left(t-1\right)\right\rangle$$

Finally, the dot product of the vector field and the derivative of the parameterization.

$$\begin{aligned}\vec{F}\left(\vec{r}\left(t\right)\right) \centerdot \vec{r}'\left(t\right) &= \left(t-1\right)^4\left(1\right) + \left(t^2-4\right)\left(2t-2\right)\\ &= \left(t-1\right)^4 + 2t^3 - 2t^2 - 8t + 8\end{aligned}$$

Make sure that you simplify the dot product with an eye towards doing the integral!

Now all we need to do is evaluate the integral.

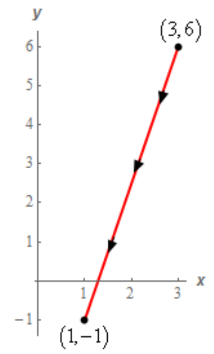
$$\begin{aligned}\int\limits_C \vec{F} \centerdot d\vec{r} &= \int_0^3 \left(t-1\right)^4 + 2t^3 - 2t^2 - 8t + 8\,dt\\ &= \left.\left[\frac{1}{5}\left(t-1\right)^5 + \frac{1}{2}t^4 - \frac{2}{3}t^3 - 4t^2 + 8t\right]\right|_0^3 = \boxed{\frac{171}{10}}\end{aligned}$$

0.0.12 Questions Solutions Line Integral with respect ds

Line Integrals - Part I - Practice Problems Solutions

1. Evaluate $\displaystyle\int_C 3x^2 - 2y\, ds$ where C is the line segment from $(3,6)$ to $(1,-1)$.

Here is a quick sketch of C with the direction specified in the problem statement shown.



Now, with the specified direction we can see that x is decreasing as we move along the curve in the specified direction. This means that we can't just determine the equation of the line and use that to work the problem. Using the equation of the line would require us to use increasing x since the limits in the integral must go from smaller to larger value.

We could of course use the fact from the notes that relates the line integral with a specified direction and the line integral with the opposite direction to allow us to use the equation of the line. However, for this problem let's just work with problem without the fact to make sure we can do that type of problem

So, we'll need to parameterize this line and we know how to parameterize the equation of a line between two points. Here is the vector form of the parameterization of the line.

$$\vec r\,(t) = (1-t)\,\langle 3, 6\rangle + t\,\langle 1, -1\rangle = \langle 3-2t, 6-7t\rangle \qquad 0 \le t \le 1$$

We could also break this up into parameter form as follows.

$$\begin{aligned} x &= 3-2t \\ y &= 6-7t \end{aligned} \qquad 0 \le t \le 1$$

Either form of the parameterization will work for the problem but we'll use the vector form for the rest of this problem.

We'll need the magnitude of the derivative of the parameterization so let's get that.

$$\vec r\,'(t) = \langle -2, -7\rangle \qquad \left\|\vec r\,'(t)\right\| = \sqrt{(-2)^2 + (-7)^2} = \sqrt{53}$$

We'll also need the integrand "evaluated" at the parameterization. Recall all this means is we replace the x/y in the integrand with the x/y from parameterization. Here is the integrand evaluated at the parameterization.

$$3x^2 - 2y = 3(3-2t)^2 - 2\,(6-7t) = 3(3-2t)^2 - 12 + 14t$$

The line integral is then,

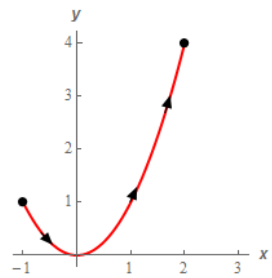
$$\begin{aligned} \int_C 3x^2 - 2y\,ds &= \int_0^1 \left(3(3-2t)^2 - 12 + 14t\right)\sqrt{53}\,dt \\ &= \sqrt{53}\left[-\frac{1}{2}(3-2t)^3 - 12t + 7t^2\right]\Bigg|_0^1 = \boxed{8\sqrt{53}} \end{aligned}$$

Note that we didn't multiply out the first term in the integrand as we could do a quick substitution to do the integral.

Section 5-2 : Line Integrals - Part I - Practice Problems Solutions

3. Evaluate $\displaystyle\int_C 6x\,ds$ where C is the portion of $y = x^2$ from $x = -1$ to $x = 2$. The direction of C is in the direction of increasing x.

Here is a quick sketch of C with the direction specified in the problem statement shown.



In this case we can just use the equation of the curve for the parameterization because the specified direction is going in the direction of increasing x which will give us integral limits from smaller value to larger value as needed. Here is a parameterization for this curve.

$$\vec r\left(t\right) = \left\langle t, t^2 \right\rangle \qquad -1 \le t \le 2$$

We could also break this up into parameter form as follows.

$$\begin{aligned} x &= t \\ y &= t^2 \end{aligned} \qquad -1 \le t \le 2$$

Either form of the parameterization will work for the problem but we'll use the vector form for the rest of this problem.

We'll need the magnitude of the derivative of the parameterization so let's get that.

$$\vec{r}\,'(t) = \langle 1, 2t \rangle \qquad \left\| \vec{r}\,'(t) \right\| = \sqrt{(1)^2 + (2t)^2} = \sqrt{1 + 4t^2}$$

We'll also need the integrand "evaluated" at the parameterization. Recall all this means is we replace the x/y in the integrand with the x/y from parameterization. Here is the integrand evaluated at the parameterization.

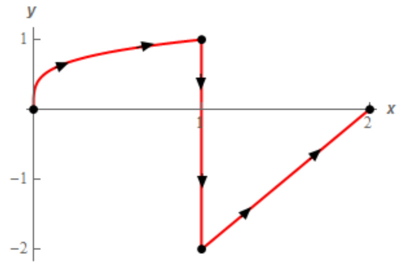
$$6x = 6t$$

The line integral is then,

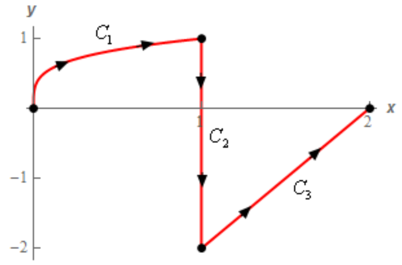
$$\int\limits_C 6x\,ds = \int_{-1}^{2} 6t\sqrt{1+4t^2}\,dt = \frac{1}{2}\left(1+4t^2\right)^{\frac{3}{2}}\Big|_{-1}^{2} = \boxed{\frac{1}{2}\left(17^{\frac{3}{2}} - 5^{\frac{3}{2}}\right)}$$

Line Integrals - Part I - Practice Problems Solutions

6. Evaluate $\int\limits_C 16y^5\, ds$ where C is the portion of $x = y^4$ from $y = 0$ to $y = 1$ followed by the line segment form $(1, 1)$ to $(1, -2)$ which in turn is followed by the line segment from $(1, -2)$ to $(2, 0)$. See the sketch below for the direction.



To help with the problem let's label each of the curves as follows,



Now let's parameterize each of these curves.

$$
\begin{aligned}
&C_1 : \vec{r}\,(t) = \left\langle t^4, t \right\rangle \qquad 0 \le t \le 1 \\
&C_2 : \vec{r}\,(t) = (1-t)\left\langle 1,1 \right\rangle + t\left\langle 1,-2 \right\rangle = \left\langle 1, 1-3t \right\rangle \qquad 0 \le t \le 1 \\
&C_3 : \vec{r}\,(t) = (1-t)\left\langle 1,-2 \right\rangle + t\left\langle 2,0 \right\rangle = \left\langle 1+t, 2t-2 \right\rangle \qquad 0 \le t \le 1
\end{aligned}
$$

For C_2 we had to use the vector form for the line segment between two points instead of the equation for the line (which is much simpler of course) because the direction was in the decreasing y direction and the limits on our integral must be from smaller to larger. We could have used the fact from the notes that tells us how the line integrals for the two directions related to allow us to use the equation of the line if we'd wanted to. We decided to do it this way just for the practice of dealing with the vector form for the line segment and it's not all that difficult to deal with the result and the limits are "nicer".

Note as well that for C_3 we could have solved for the equation of the line and used that because the direction is in the increasing x direction. However, the vector form for the line segment between two points is just as easy to use so we used that instead.

Okay, we now need to compute the line integral along each of these curves. Unlike the first few problems in this section where we found the magnitude and the integrand prior to the integration step we're just going to just straight into the integral and do all the work there.

Here is the integral along each of the curves.

$$
\int\limits_{C_1} 16y^5\,ds = \int_0^1 16(t)^5 \sqrt{\left(4t^3\right)^2 + (1)^2}\,dt = \int_0^1 16t^5 \sqrt{16t^6+1}\,dt
$$

$$= \frac{1}{9}\left(16t^6+1\right)^{\frac{3}{2}}\Big|_0^1 = \underline{\frac{1}{9}\left(17^{\frac{3}{2}}-1\right) = 7.6770}$$

$$\int\limits_{C_2} 16y^5\, ds = \int_0^1 16(1-3t)^5\sqrt{(0)^2+(-3)^2}\,dt = \int_0^1 48(1-3t)^5\, dt$$

$$= -\frac{8}{3}(1-3t)^6\Big|_0^1 = \underline{-168}$$

$$\int\limits_{C_3} 16y^5\, ds = \int_0^1 16(2t-2)^5\sqrt{(1)^2+(2)^2}\,dt = \int_0^1 16\sqrt{5}(2t-2)^5\, dt$$

$$= \frac{4\sqrt{5}}{3}(2t-2)^6\Big|_0^1 = \underline{-\frac{256\sqrt{5}}{3} = -190.8111}$$

Okay to finish this problem out all we need to do is add up the line integrals over these curves to get the full line integral.

$$\int\limits_{C} 16y^5\, ds = \left(\frac{1}{9}\left(17^{\frac{3}{2}}+1\right)\right) + (-168) + \left(-\frac{256\sqrt{5}}{3}\right) = \boxed{-351.1341}$$

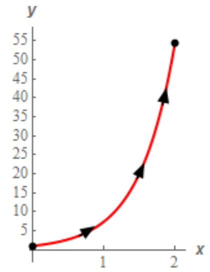
Note that we put parenthesis around the result of each individual line integral simply to illustrate where it came from and they aren't needed in general of course.

0.0.13 Questions Solutions Line Integral with respect dxdy

Line Integrals - Part II - Practice Problems Solutions

1. Evaluate $\displaystyle\int\limits_C \sqrt{1+y}\,dy$ where C is the portion of $y = \mathbf{e}^{2x}$ from $x = 0$ to $x = 2$.

Here is a quick sketch of C with the direction specified in the problem statement shown.



Next, we need to parameterize the curve.

$$\vec r\,(t) = \left\langle t, \mathbf{e}^{2t} \right\rangle \qquad 0 \le t \le 2$$

Now we need to evaluate the line integral. Be careful with this type line integral. Note that the differential, in this case, is dy and not ds as they were in the previous section.

All we need to do is recall that $dy = y'\,dt$ when we convert the line integral into a "standard" integral.

So, let's evaluate the line integral. Just remember to "plug in" the parameterization into the integrand (*i.e.* replace the x and y in the integrand with the x and y components of the parameterization) and to convert the differential properly.

Here is the line integral.

$$\int\limits_C \sqrt{1+y}\,dy = \int_0^2 \sqrt{1+\mathbf{e}^{2t}}\,\left(2\mathbf{e}^{2t}\right)\,dt$$

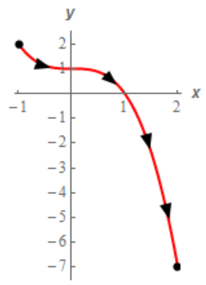
$$= \left[\frac{2}{3}\left(1+\mathbf{e}^{2t}\right)^{\frac{3}{2}}\right]\Bigg|_0^2 = \boxed{\frac{2}{3}\left[\left(1+\mathbf{e}^{4}\right)^{\frac{3}{2}} - 2^{\frac{3}{2}}\right] = 274.4897}$$

Note that, in this case, the integral ended up being a simple substitution.

Line Integrals - Part II - Practice Problems Solutions

2. Evaluate $\int\limits_C 2y\,dx + (1-x)\;dy$ where C is portion of $y = 1 - x^3$ from $x = -1$ to $x = 2$.

Here is a quick sketch of C with the direction specified in the problem statement shown.



Next, we need to parameterize the curve.

$$\vec r\,(t) = \langle t, 1 - t^3\rangle \qquad -1 \le t \le 2$$

Now we need to evaluate the line integral. Be careful with this type of line integral. In this case we have both a dx and a dy in the integrand. Recall that this is just a simplified notation for,

$$\int\limits_C 2y\,dx + (1-x)\;dy = \int\limits_C 2y\,dx + \int\limits_C 1 - x\,dy$$

Then all we need to do is recall that $dx = x'\,dt$ and $dy = y'\,dt$ when we convert the line integral into a "standard" integral.

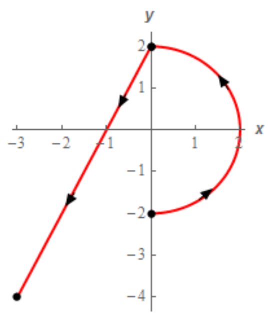
So, let's evaluate the line integral. Just remember to "plug in" the parameterization into the integrand (*i.e.* replace the x and y in the integrand with the x and y components of the parameterization) and to convert the differentials properly.

Here is the line integral.

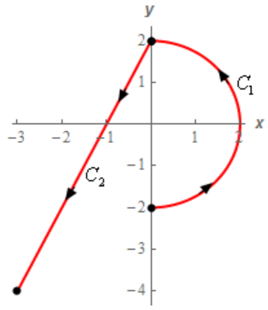
$$\begin{aligned}\int\limits_C 2y\,dx + (1-x)\;dy &= \int\limits_C 2y\,dx + \int\limits_C 1 - x\,dy \\ &= \int_{-1}^{2} 2\left(1-t^3\right)\;(1)\;dt + \int_{-1}^{2} (1-t)\;\left(-3t^2\right)\;dt \\ &= \int_{-1}^{2} 2\left(1-t^3\right)\;dt - 3\int_{-1}^{2} t^2 - t^3\,dt \\ &= \int_{-1}^{2} t^3 - 3t^2 + 2\,dt \\ &= \left.\left[\frac{1}{4}t^4 - t^3 + 2t\right]\right|_{-1}^{2} = \boxed{\frac{3}{4}}\end{aligned}$$

Note that, in this case, we combined the two integrals into a single integral prior to actually evaluating the integral. This doesn't need to be done but can, on occasion, simplify the integrand and hence the evaluation of the integral.

4. Evaluate $\int\limits_C 1 + x^3\,dx$ where C is the right half of the circle of radius 2 with counter clockwise rotation followed by the line segment from $(0,2)$ to $(-3,-4)$. See the sketch below for the direction.



To help with the problem let's label each of the curves as follows,



Now let's parameterize each of these curves.

$C_1: \ \vec{r}\,(t) = \langle 2\cos(t), 2\sin(t)\rangle \qquad -\frac{1}{2}\pi \le t \le \frac{1}{2}\pi$

$C_2: \vec{r}\ (t) = (1-t)\,\langle 0, 2\rangle + t\,\langle -3, -4\rangle = \langle -3t, 2-6t\rangle \qquad 0 \le t \le 1$

Now we need to compute the line integral for each of the curves.

$$\int_{C_1} 1 + x^3\,dx = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \left[1 + (2\cos(t))^3\right](-2\sin(t))\ dt$$

$$= \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} -2\sin(t) - 16\cos^3(t)\sin(t)\,dt$$

$$= \left(2\cos(t) + 4\cos^4(t)\right)\Big|_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} = \underline{0}$$

$$\int_{C_2} 1 + x^3\,dx = \int_0^1 \left[1 + (-3t)^3\right](-3)\ dt$$

$$= \int_0^1 -3 + 81t^3\,dt$$

$$= \left(-3t + \frac{81}{4}t^4\right)\Big|_0^1 = \underline{\frac{69}{4}}$$

Don't forget to correctly deal with the differentials when converting the line integral into a "standard" integral.

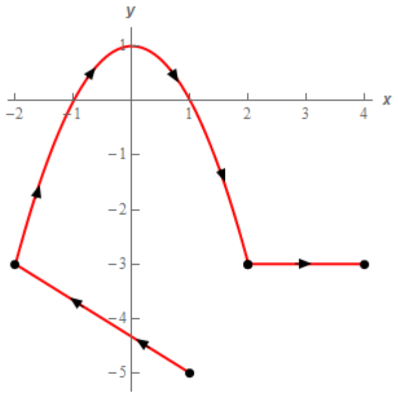
Okay to finish this problem out all we need to do is add up the line integrals over these curves to get the full line integral.

$$\int_C 1 + x^3 \, dx = (0) + \left(\frac{69}{4}\right) = \boxed{\frac{69}{4}}$$

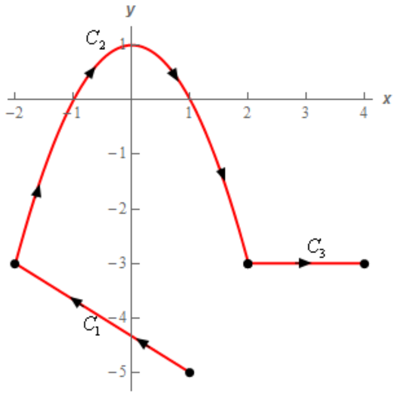
Note that we put parenthesis around the result of each individual line integral simply to illustrate where it came from and they aren't needed in general of course.

Line Integrals - Part II - Practice Problems Solutions

5. Evaluate $\displaystyle\int\limits_C 2x^2\,dy - xy\,dx$ where C is the line segment from $(1,-5)$ to $(-2,-3)$ followed by the portion of $y = 1 - x^2$ from $x = -2$ to $x = 2$ which in turn is followed by the line segment from $(2,-3)$ to $(4,-3)$. See the sketch below for the direction.



To help with the problem let's label each of the curves as follows,



Now let's parameterize each of these curves.

$$C_1: \ \vec{r}(t) = (1-t)\langle 1,-5\rangle + t\langle -2,-3\rangle = \langle 1-3t, -5+2t\rangle \qquad 0 \le t \le 1$$
$$C_2: \ \vec{r}(t) = \left\langle t, 1-t^2\right\rangle \qquad -2 \le t \le 2$$
$$C_3: \ \vec{r}(t) = \langle t,-3\rangle \qquad 2 \le t \le 4$$

Note that for C_1 we had to use the vector form for the line segment between two points because the specified direction was in the decreasing x direction and so the equation of the line wouldn't work since the limits of the line integral need to go from smaller to larger values.

We did just use the equation of the line for C_3 since it was simple enough to do and the limits were also nice enough.

Now we need to compute the line integral for each of the curves.

$$\int 2x^2\,dy - xy\,dx = \int 2x^2\,dy - \int xy\,dx$$

$C_1 \qquad C_1 \qquad C_1$

$$= \int_0^1 2(1-3t)^2\,(2)\;dt - \int_0^1 (1-3t)\,(-5+2t)\,(-3)\;dt$$

$$= \int_0^1 4(1-3t)^2 - 3\left(6t^2 - 17t + 5\right)\;dt$$

$$= \left(-\frac{4}{9}(1-3t)^3 - 3\left(2t^3 - \frac{17}{2}t^2 + 5t\right)\right)\Bigg|_0^1 = \underline{\frac{17}{2}}$$

$$\int_{C_2} 2x^2\,dy - xy\,dx = \int_{C_2} 2x^2\,dy - \int_{C_2} xy\,dx$$

$$= \int_{-2}^{2} 2(t)^2\,(-2t)\;dt - \int_{-2}^{2} (t)\left(1-t^2\right)(1)\;dt$$

$$= \int_{-2}^{2} -3t^3 - t\,dt$$

$$= \left(-\frac{3}{4}t^4 - \frac{1}{2}t^2\right)\Bigg|_{-2}^{2} = \underline{0}$$

$$\int_{C_3} 2x^2\,dy - xy\,dx = \int_{C_3} 2x^2\,dy - \int_{C_2} xy\,dx$$

$$= \int_{2}^{4} 2(t)^2\,(0)\;dt - \int_{2}^{4} (t)\,(-3)\,(1)\;dt$$

$$= \int_{2}^{4} 3t\,dt$$

$$= \left(\frac{3}{2}t^2\right)\Bigg|_{2}^{4} = \underline{18}$$

Don’t forget to correctly deal with the differentials when converting the line integral into a “standard” integral.

Also, don’t get excited when one of the differentials “evaluates” to zero as the first one did in the C_3 integral. That will happen on occasion and is not something to get worried about when it does.

Okay to finish this problem out all we need to do is add up the line integrals over these curves to get the full line integral.

$$\int\limits_C 2x^2\,dy - xy\,dx = \left(\frac{17}{2}\right) + (0) + (18) = \boxed{\frac{53}{2}}$$

Note that we put parenthesis around the result of each individual line integral simply to illustrate where it came from and they aren't needed in general of course.

Line Integrals - Part II - Practice Problems Solutions

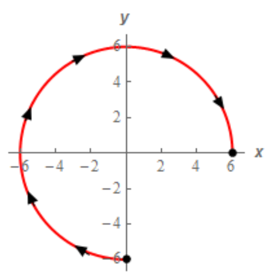
6. Evaluate $\displaystyle\int\limits_C (x-y)\,dx - yx^2\,dy$ for each of the following curves.

(a) C is the portion of the circle of radius 6 in the 1st, 2nd and 3rd quadrant with clockwise rotation.

(b) C is the line segment from $(0,-6)$ to $(6,0)$.

(a) C is the portion of the circle of radius 6 in the 1st, 2nd and 3rd quadrant with clockwise rotation.

Let's start off with a quick sketch of the curve for this part of the problem.



Here is the parameterization for this curve.

$$C: \vec{r}\,(t) = \langle 6\cos(t), -6\sin(t)\rangle \qquad \frac{1}{\;}\pi < t < 2\pi$$

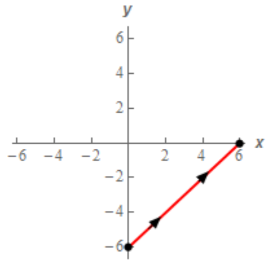
Here is the line integral for this curve.

$$\begin{aligned}
\int\limits_C (x-y)\,dx - yx^2\,dy &= \int\limits_C (x-y)\,dx - \int\limits_C yx^2\,dy \\
&= \int_{\frac{1}{2}\pi}^{2\pi} \left[6\cos(t) - (-6\sin(t))\right](-6\sin(t))\,dt \\
&\qquad - \int_{\frac{1}{2}\pi}^{2\pi} \left[(6\sin(t))(6\cos(t))^2\right](-6\cos(t))\,dt \\
&= \int_{\frac{1}{2}\pi}^{2\pi} -36\cos(t)\sin(t) - 36\sin^2(t) - 1296\sin(t)\cos^3(t)\,dt \\
&= \int_{\frac{1}{2}\pi}^{2\pi} -18\sin(2t) - 18\left(1-\cos(2t)\right) - 1296\sin(t)\cos^3(t)\,dt \\
&= \left(9\cos(2t) - 18t + 9\sin(2t) + 324\cos^4(t)\right)\Big|_{\frac{1}{2}\pi}^{2\pi} \\
&= \boxed{342 - 27\pi = 257.1770}
\end{aligned}$$

Don't forget to correctly deal with the differentials when converting the line integral into a "standard" integral.

(b) C is the line segment from $(0,-6)$ to $(6,0)$.

Let's start off with a quick sketch of the curve for this part of the problem.



So, what we have in this part is a different curve that goes from $(0,-6)$ to $(6,0)$. Despite the fact that this curve has the same starting and ending point as the curve in the first part there is no reason to expect the line integral to have the same value. Therefore we'll need to go through the work and see what we get from the line integral.

We'll need to parameterize the curve so let's take care of that.

$$C: \ \vec{r}(t) = (1-t)\langle 0,-6\rangle + t\langle 6,0\rangle = \langle 6t, -6+6t\rangle \qquad 0 \le t \le 1$$

Note that we could have just found the equation of this curve but it seemed just as easy to just use the vector form of the line segment between two points.

Now all we need to do is compute the line integral.

$$\begin{aligned}\int_C (x-y)\,dx - yx^2\,dy &= \int_C (x-y)\,dx - \int_C yx^2\,dy \\ &= \int_0^1 [6t-(-6+6t)](6)\,dt - \int_0^1 \left[(-6+6t)(6t)^2\right](6)\,dt \\ &= \int_0^1 36 + 1296t^2 - 1296t^3\,dt \\ &= \left(36t + 432t^3 - 324t^4\right)\Big|_0^1 = \boxed{144}\end{aligned}$$

So, as noted at the start of this part the value of the line integral was not the same as the value of the line integral in the first part despite the same starting and ending points for the curve. Note that it is possible for two line integrals with the same starting and ending points to have the same value but we can't expect that to happen and so need to go through and do the work.

0.0.14 Question Solutions Spherical

Figure 11 shows how E is swept out if we integrate first with respect to ρ, then ϕ, and then θ. The volume of E is

$$V(E) = \iiint_E dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\phi \left[\frac{\rho^3}{3} \right]_{\rho=0}^{\rho=\cos\phi} d\phi$$

$$= \frac{2\pi}{3} \int_0^{\pi/4} \sin\phi \cos^3\phi \, d\phi = \frac{2\pi}{3} \left[-\frac{\cos^4\phi}{4} \right]_0^{\pi/4} = \frac{\pi}{8}$$

TEC Visual 15.9 shows an animation of Figure 11.

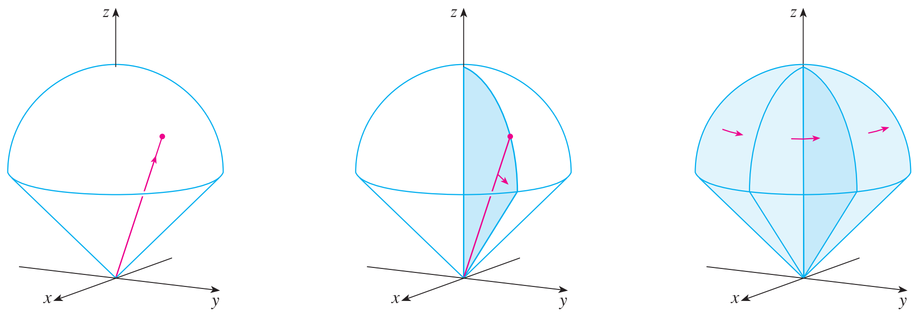


FIGURE 11

ρ varies from 0 to $\cos\phi$ while ϕ and θ arec onstant.

ϕ varies from 0 to $\pi/4$ while θ is constant.

θ varies from 0 to 2π.

15.9 Exercises

1–2 Plot the point whose spherical coordinates are given. Then find the rectangular coordinates of the point.

1. (a) $(6, \pi/3, \pi/6)$ (b) $(3, \pi/2, 3\pi/4)$

2. (a) $(2, \pi/2, \pi/2)$ (b) $(4, -\pi/4, \pi/3)$

3–4 Change from rectangular to spherical coordinates.

3. (a) $(0, -2, 0)$ (b) $(-1, 1, -\sqrt{2})$

4. (a) $(1, 0, \sqrt{3})$ (b) $(\sqrt{3}, -1, 2\sqrt{3})$

5–6 Describe in words the surface whose equation is given.

5. $\phi = \pi/3$ **6.** $\rho = 3$

7–8 Identify the surface whose equation is given.

7. $\rho = \sin\theta \sin\phi$ **8.** $\rho^2(\sin^2\phi \sin^2\theta + \cos^2\phi) = 9$

9–10 Write the equation in spherical coordinates.

9. (a) $z^2 = x^2 + y^2$ (b) $x^2 + z^2 = 9$

10. (a) $x^2 - 2x + y^2 + z^2 = 0$ (b) $x + 2y + 3z = 1$

Graphing calculator or computer required CAS Computer algebra sy angle of $\pi/6$.

21–34 Use spherical coordinates.

21. Evaluate $\iiint_B (x^2 + y^2 + z^2)^2 \, dV$, where B is the ball with center the origin and radius 5.

22. Evaluate $\iiint_H (9 - x^2 - y^2) \, dV$, where H is the solid hemisphere $x^2 + y^2 + z^2 \leqslant 9$, $z \geqslant 0$.

23. Evaluate $\iiint_E (x^2 + y^2) \, dV$, where E lies between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 9$.

24. Evaluate $\iiint_E y^2 \, dV$, where E is the solid hemisphere $x^2 + y^2 + z^2 \leqslant 9$, $y \geqslant 0$.

25. Evaluate $\iiint_E xe^{x^2+y^2+z^2} \, dV$, where E is the portion of the unit ball $x^2 + y^2 + z^2 \leqslant 1$ that lies in the first octant.

26. Evaluate $\iiint_E xyz \, dV$, where E lies between the spheres $\rho = 2$ and $\rho = 4$ and above the cone $\phi = \pi/3$.

27. Find the volume of the part of the ball $\rho \leqslant a$ that lies between the cones $\phi = \pi/6$ and $\phi = \pi/3$.

28. Find the average distance from a point in a ball of radius a to its center.

29. (a) Find the volume of the solid that lies above the cone $\phi = \pi/3$ and below the sphere $\rho = 4 \cos \phi$.
(b) Find the centroid of the solid in part (a).

30. Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$, above the xy-plane, and below the cone $z = \sqrt{x^2 + y^2}$.

31. (a) Find the centroid of the solid in Example 4.
(b) Find the moment of inertia about the z-axis for this solid.

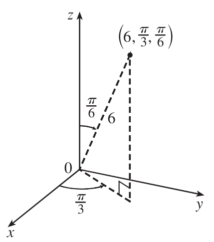
We split the region of integration where the outside boundary changes from the vertical line $x = 1$ to the circle $x^2 + y^2 = a^2$ or $r = 1$. R_1 is a right triangle, so $\cos\theta = \frac{1}{a}$. Thus, the boundary between R_1 and R_2 is $\theta = \cos^{-1}\left(\frac{1}{a}\right)$ in polar coordinates, or $y = \sqrt{a^2 - 1}\,x$ in rectangular coordinates. Using rectangular coordinates for the region R_1 and polar coordinates for R_2, we find the total volume of the solid to be

$$V = 16\left[\int_0^1 \int_0^{\sqrt{a^2-1}\,x} \sqrt{1 - x^2}\,dy\,dx + \int_{\cos^{-1}(1/a)}^{\pi/4} \int_0^a \sqrt{1 - r^2\cos^2\theta}\,r\,dr\,d\theta\right]$$

If $a \geq \sqrt{2}$, the cylinder $x^2 + y^2 = 1$ completely encloses the intersection of the other two cylinders, so the solid of intersection of the three cylinders coincides with the intersection of $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$ as illustrated in Exercise 15.6.24. Its volume is $V = 16\int_0^1 \int_0^x \sqrt{1 - x^2}\,dy\,dx$.

15.9 Triple Integrals in Spherical Coordinates

1. (a)

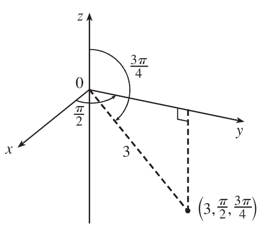


From Equations 1, $x = \rho\sin\phi\cos\theta = 6\sin\frac{\pi}{6}\cos\frac{\pi}{3} = 6 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{2}$,

$y = \rho\sin\phi\sin\theta = 6\sin\frac{\pi}{6}\sin\frac{\pi}{3} = 6 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$, and

$z = \rho\cos\phi = 6\cos\frac{\pi}{6} = 6 \cdot \frac{\sqrt{3}}{2} = 3\sqrt{3}$, so the point is $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}, 3\sqrt{3}\right)$ in rectangular coordinates.

(b)

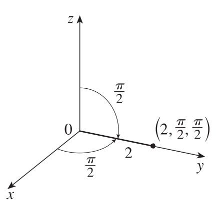


$x = 3\sin\frac{3\pi}{4}\cos\frac{\pi}{2} = 3 \cdot \frac{\sqrt{2}}{2} \cdot 0 = 0$,

$y = 3\sin\frac{3\pi}{4}\sin\frac{\pi}{2} = 3 \cdot \frac{\sqrt{2}}{2} \cdot 1 = \frac{3\sqrt{2}}{2}$, and

$z = 3\cos\frac{3\pi}{4} = 3\left(-\frac{\sqrt{2}}{2}\right) = -\frac{3\sqrt{2}}{2}$, so the point is $\left(0, \frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$ in rectangular coordinates.

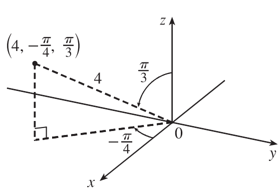
2. (a)



$x = 2\sin\frac{\pi}{2}\cos\frac{\pi}{2} = 2 \cdot 1 \cdot 0 = 0$, $y = 2\sin\frac{\pi}{2}\sin\frac{\pi}{2} = 2 \cdot 1 \cdot 1 = 2$,

$z = 2\cos\frac{\pi}{2} = 2 \cdot 0 = 0$ so the point is $(0, 2, 0)$ in rectangular coordinates.

(b)



$x = 4\sin\frac{\pi}{3}\cos\left(-\frac{\pi}{4}\right) = 4 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \sqrt{6}$,

$y = 4\sin\frac{\pi}{3}\sin\left(-\frac{\pi}{4}\right) = 4\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) = -\sqrt{6}$,

$z = 4\cos\frac{\pi}{3} = 4 \cdot \frac{1}{2} = 2$ so the point is $\left(\sqrt{6}, -\sqrt{6}, 2\right)$ in rectangular coordinates.

3. (a) From Equations 1 and 2, $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{0^2+(-2)^2+0^2} = 2$, $\cos\phi = \dfrac{z}{\rho} = \dfrac{0}{2} = 0 \;\Rightarrow\; \phi = \dfrac{\pi}{2}$, and

$\cos\theta = \dfrac{x}{\rho\sin\phi} = \dfrac{0}{2\sin(\pi/2)} = 0 \;\Rightarrow\; \theta = \dfrac{3\pi}{2}$ [since $y<0$]. Thus spherical coordinates are $\left(2, \dfrac{3\pi}{2}, \dfrac{\pi}{2}\right)$.

(b) $\rho = \sqrt{1+1+2} = 2$, $\cos\phi = \dfrac{z}{\rho} = \dfrac{-\sqrt{2}}{2} \;\Rightarrow\; \phi = \dfrac{3\pi}{4}$, and

$\cos\theta = \dfrac{x}{\rho\sin\phi} = \dfrac{-1}{2\sin(3\pi/4)} = \dfrac{-1}{2\left(\sqrt{2}/2\right)} = -\dfrac{1}{\sqrt{2}} \;\Rightarrow\; \theta = \dfrac{3\pi}{4}$ [since $y>0$]. Thus spherical coordinates are $\left(2, \dfrac{3\pi}{4}, \dfrac{3\pi}{4}\right)$.

4. (a) $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{1+0+3} = 2$, $\cos\phi = \dfrac{z}{\rho} = \dfrac{\sqrt{3}}{2} \;\Rightarrow\; \phi = \dfrac{\pi}{6}$, and $\cos\theta = \dfrac{x}{\rho\sin\phi} = \dfrac{1}{2\sin(\pi/6)} = 1 \;\Rightarrow$ $\theta = 0$. Thus spherical coordinates are $\left(2, 0, \dfrac{\pi}{6}\right)$.

(b) $\rho = \sqrt{3+1+12} = 4$, $\cos\phi = \dfrac{z}{\rho} = \dfrac{2\sqrt{3}}{4} = \dfrac{\sqrt{3}}{2} \;\Rightarrow\; \phi = \dfrac{\pi}{6}$, and $\cos\theta = \dfrac{x}{\rho\sin\phi} = \dfrac{\sqrt{3}}{4\sin(\pi/6)} = \dfrac{\sqrt{3}}{2} \;\Rightarrow$ $\theta = \dfrac{11\pi}{6}$ [since $y<0$]. Thus spherical coordinates are $\left(4, \dfrac{11\pi}{6}, \dfrac{\pi}{6}\right)$.

5. Since $\phi = \frac{\pi}{3}$, the surface is the top half of the right circular cone with vertex at the origin and axis the positive z-axis.

6. Since $\rho = 3$, $x^2+y^2+z^2 = 9$ and the surface is a sphere with center the origin and radius 3.

7. $\rho = \sin\theta\sin\phi \;\Rightarrow\; \rho^2 = \rho\sin\theta\sin\phi \;\Leftrightarrow\; x^2+y^2+z^2 = y \;\Leftrightarrow\; x^2+y^2-y+\frac{1}{4}+z^2 = \frac{1}{4} \;\Leftrightarrow$ $x^2+\left(y-\frac{1}{2}\right)^2+z^2 = \frac{1}{4}$. Therefore, the surface is a sphere of radius $\frac{1}{2}$ centered at $\left(0, \frac{1}{2}, 0\right)$.

8. $\rho^2\left(\sin^2\phi\sin^2\theta+\cos^2\phi\right) = 9 \;\Leftrightarrow\; (\rho\sin\phi\sin\theta)^2+(\rho\cos\phi)^2 = 9 \;\Leftrightarrow\; y^2+z^2 = 9$. Thus the surface is a circular cylinder of radius 3 with axis the x-axis.

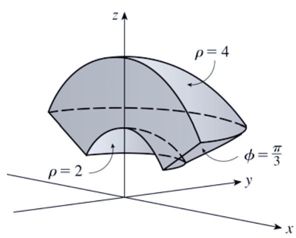
9. (a) $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$, so the equation $z^2 = x^2+y^2$ becomes $(\rho\cos\phi)^2 = (\rho\sin\phi\cos\theta)^2+(\rho\sin\phi\sin\theta)^2$ or $\rho^2\cos^2\phi = \rho^2\sin^2\phi$. If $\rho\neq 0$, this becomes $\cos^2\phi = \sin^2\phi$. ($\rho = 0$ corresponds to the origin which is included in the surface.) There are many equivalent equations in spherical coordinates, such as $\tan^2\phi = 1$, $2\cos^2\phi = 1$, $\cos 2\phi = 0$, or even $\phi = \frac{\pi}{4}$, $\phi = \frac{3\pi}{4}$.

(b) $x^2+z^2 = 9 \;\Leftrightarrow\; (\rho\sin\phi\cos\theta)^2+(\rho\cos\phi)^2 = 9 \;\Leftrightarrow\; \rho^2\sin^2\phi\cos^2\theta+\rho^2\cos^2\phi = 9$ or $\rho^2\left(\sin^2\phi\cos^2\theta+\cos^2\phi\right) = 9$.

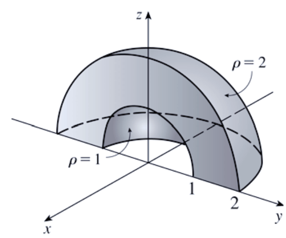
10. (a) $x^2-2x+y^2+z^2 = 0 \;\Leftrightarrow\; \left(x^2+y^2+z^2\right)-2x = 0 \;\Leftrightarrow\; \rho^2-2\left(\rho\sin\phi\cos\theta\right) = 0$ or $\rho = 2\sin\phi\cos\theta$.

(b) $x+2y+3z = 1 \;\Leftrightarrow\; \rho\sin\phi\cos\theta+2\rho\sin\phi\sin\theta+3\rho\cos\phi = 1$ or $\rho = 1/\left(\sin\phi\cos\theta+2\sin\phi\sin\theta+3\cos\phi\right)$.

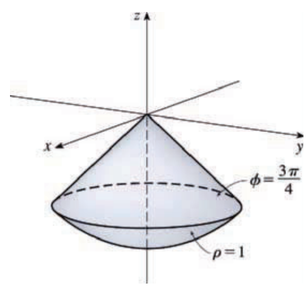
11. $2 \le \rho \le 4$ represents the solid region between and including the spheres of radii 2 and 4, centered at the origin. $0 \le \phi \le \frac{\pi}{3}$ restricts the solid to that portion on or above the cone $\phi = \frac{\pi}{3}$, and $0 \le \theta \le \pi$ further restricts the solid to that portion on or to the right of the xz-plane.



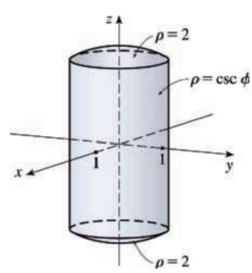
12. $1 \le \rho \le 2$ represents the solid region between and including the spheres of radii 1 and 2, centered at the origin. $0 \le \phi \le \frac{\pi}{2}$ restricts the solid to that portion on or above the xy-plane, and $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$ further restricts the solid to that portion on or behind the yz-plane.



13. $\rho \le 1$ represents the solid sphere of radius 1 centered at the origin. $\frac{3\pi}{4} \le \phi \le \pi$ restricts the solid to that portion on or below the cone $\phi = \frac{3\pi}{4}$.



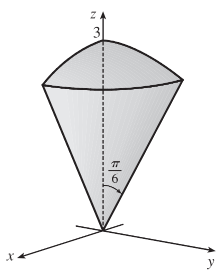
14. $\rho \le 2$ represents the solid sphere of radius 2 centered at the origin. Notice that $x^2 + y^2 = (\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2 = \rho^2 \sin^2\phi$. Then $\rho = \csc\phi \;\Rightarrow\; \rho \sin\phi = 1 \;\Rightarrow\; \rho^2 \sin^2\phi = x^2 + y^2 = 1$, so $\rho \le \csc\phi$ restricts the solid to that portion on or inside the circular cylinder $x^2 + y^2 = 1$.



15. $z \ge \sqrt{x^2 + y^2}$ because the solid lies above the cone. Squaring both sides of this inequality gives $z^2 \ge x^2 + y^2 \;\Rightarrow\; 2z^2 \ge x^2 + y^2 + z^2 = \rho^2 \;\Rightarrow\; z^2 = \rho^2 \cos^2\phi \ge \frac{1}{2}\rho^2 \;\Rightarrow\; \cos^2\phi \ge \frac{1}{2}$. The cone opens upward so that the inequality is $\cos\phi \ge \frac{1}{\sqrt{2}}$, or equivalently $0 \le \phi \le \frac{\pi}{4}$. In spherical coordinates the sphere $z = x^2 + y^2 + z^2$ is $\rho\cos\phi = \rho^2 \;\Rightarrow\; \rho = \cos\phi$. $0 \le \rho \le \cos\phi$ because the solid lies below the sphere. The solid can therefore be described as the region in spherical coordinates satisfying $0 \le \rho \le \cos\phi$, $0 \le \phi \le \frac{\pi}{4}$.

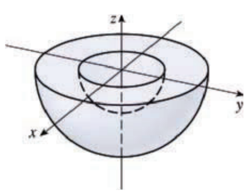
16. (a) The hollow ball is a spherical shell with outer radius 15 cm and inner radius 14.5 cm. If we center the ball at the origin of the coordinate system and use centimeters as the unit of measurement, then spherical coordinates conveniently describe the hollow ball as $14.5 \le \rho \le 15, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi$.

(b) If we position the ball as in part (a), one possibility is to take the half of the ball that is above the xy-plane which is described by $14.5 \le \rho \le 15, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi/2$.

17. 

The region of integration is given in spherical coordinates by $E = \{(\rho, \theta, \phi) \mid 0 \le \rho \le 3, 0 \le \theta \le \pi/2, 0 \le \phi \le \pi/6\}$. This represents the solid region in the first octant bounded above by the sphere $\rho = 3$ and below by the cone $\phi = \pi/6$.

$$\begin{aligned}\int_0^{\pi/6}\int_0^{\pi/2}\int_0^3 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi &= \int_0^{\pi/6} \sin\phi \, d\phi \int_0^{\pi/2} d\theta \int_0^3 \rho^2 \, d\rho \\ &= \left[-\cos\phi\right]_0^{\pi/6} \left[\,\theta\,\right]_0^{\pi/2} \left[\tfrac{1}{3}\rho^3\right]_0^3 \\ &= \left(1 - \frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{2}\right)(9) = \frac{9\pi}{4}\left(2 - \sqrt{3}\right)\end{aligned}$$

18. 

The region of integration is given in spherical coordinates by $E = \{(\rho, \theta, \phi) \mid 1 \le \rho \le 2, 0 \le \theta \le 2\pi, \pi/2 \le \phi \le \pi\}$. This represents the solid region between the spheres $\rho = 1$ and $\rho = 2$ and below the xy-plane.

$$\begin{aligned}\int_0^{2\pi}\int_{\pi/2}^{\pi}\int_1^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta &= \int_0^{2\pi} d\theta \int_{\pi/2}^{\pi} \sin\phi \, d\phi \int_1^2 \rho^2 \, d\rho \\ &= \left[\,\theta\,\right]_0^{2\pi} \left[-\cos\phi\right]_{\pi/2}^{\pi} \left[\tfrac{1}{3}\rho^3\right]_1^2 \\ &= 2\pi(1)\left(\tfrac{7}{3}\right) = \tfrac{14\pi}{3}\end{aligned}$$

19. The solid E is most conveniently described if we use cylindrical coordinates:
$E = \left\{(r, \theta, z) \mid 0 \le \theta \le \frac{\pi}{2}, 0 \le r \le 3, 0 \le z \le 2\right\}$. Then
$\iiint_E f(x, y, z)\, dV = \int_0^{\pi/2}\int_0^3\int_0^2 f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$.

20. The solid E is most conveniently described if we use spherical coordinates:
$E = \left\{(\rho, \theta, \phi) \mid 1 \le \rho \le 2, \frac{\pi}{2} \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{2}\right\}$. Then
$\iiint_E f(x, y, z)\, dV = \int_0^{\pi/2}\int_{\pi/2}^{2\pi}\int_1^2 f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$.

21. In spherical coordinates, B is represented by $\{(\rho, \theta, \phi) \mid 0 \le \rho \le 5, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$. Thus

$$\begin{aligned}\iiint_B (x^2 + y^2 + z^2)^2\, dV &= \int_0^{\pi}\int_0^{2\pi}\int_0^5 (\rho^2)^2 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi = \int_0^{\pi} \sin\phi \, d\phi \int_0^{2\pi} d\theta \int_0^5 \rho^6 \, d\rho \\ &= \left[-\cos\phi\right]_0^{\pi} \left[\,\theta\,\right]_0^{2\pi} \left[\tfrac{1}{7}\rho^7\right]_0^5 = (2)(2\pi)\left(\tfrac{78{,}125}{7}\right) \\ &= \tfrac{312{,}500}{7}\pi \approx 140{,}249.7\end{aligned}$$

22. In spherical coordinates, H is represented by $\left\{(\rho,\theta,\phi)\,\middle|\,0\le\rho\le 3, 0\le\theta\le 2\pi, 0\le\phi\le\frac{\pi}{2}\right\}$. Thus

$$\begin{aligned}\iiint_H (9-x^2-y^2)\,dV &= \int_0^{\pi/2}\int_0^{2\pi}\int_0^3 \left[9-(\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta)\right]\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi\\ &= \int_0^{\pi/2}\int_0^{2\pi}\int_0^3 (9-\rho^2\sin^2\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi\\ &= \int_0^{\pi/2}\int_0^{2\pi}\left[3\rho^3-\tfrac{1}{5}\rho^5\sin^2\phi\right]_{\rho=0}^{\rho=3}\sin\phi\,d\rho\,d\theta\,d\phi\\ &= \int_0^{\pi/2}\int_0^{2\pi}\left(81\sin\phi-\tfrac{243}{5}\sin^3\phi\right)d\theta\,d\phi\\ &= \int_0^{2\pi}d\theta\int_0^{\pi/2}\left[81\sin\phi-\tfrac{243}{5}(1-\cos^2\phi)\sin\phi\right]d\phi\\ &= 2\pi\left[-81\cos\phi-\tfrac{243}{5}\left(\tfrac{1}{3}\cos^3\phi-\cos\phi\right)\right]_0^{\pi/2}\\ &= 2\pi\left[0+81+\tfrac{243}{5}\left(-\tfrac{2}{3}\right)\right]=\tfrac{486}{5}\pi\end{aligned}$$

23. In spherical coordinates, E is represented by $\{(\rho,\theta,\phi)\mid 2\le\rho\le 3, 0\le\theta\le 2\pi, 0\le\phi\le\pi\}$ and

$x^2+y^2=\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta=\rho^2\sin^2\phi\left(\cos^2\theta+\sin^2\theta\right)=\rho^2\sin^2\phi$. Thus

$$\begin{aligned}\iiint_E (x^2+y^2)\,dV &= \int_0^{\pi}\int_0^{2\pi}\int_2^3(\rho^2\sin^2\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi=\int_0^{\pi}\sin^3\phi\,d\phi\int_0^{2\pi}d\theta\int_2^3\rho^4\,d\rho\\ &= \int_0^{\pi}(1-\cos^2\phi)\,\sin\phi\,d\phi\,\left[\,\theta\,\right]_0^{2\pi}\left[\tfrac{1}{5}\rho^5\right]_2^3=\left[-\cos\phi+\tfrac{1}{3}\cos^3\phi\right]_0^{\pi}(2\pi)\cdot\tfrac{1}{5}(243-32)\\ &= \left(1-\tfrac{1}{3}+1-\tfrac{1}{3}\right)(2\pi)\left(\tfrac{211}{5}\right)=\tfrac{1688\pi}{15}\end{aligned}$$

24. In spherical coordinates, E is represented by $\{(\rho,\theta,\phi)\mid 0\le\rho\le 3, 0\le\theta\le\pi, 0\le\phi\le\pi\}$. Thus

$$\begin{aligned}\iiint_E y^2\,dV &= \int_0^{\pi}\int_0^{\pi}\int_0^3(\rho\sin\phi\sin\theta)^2\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi=\int_0^{\pi}\sin^3\phi\,d\phi\int_0^{\pi}\sin^2\theta\,d\theta\int_0^3\rho^4\,d\rho\\ &= \int_0^{\pi}(1-\cos^2\phi)\,\sin\phi\,d\phi\int_0^{\pi}\tfrac{1}{2}(1-\cos 2\theta)\,d\theta\int_0^3\rho^4\,d\rho\\ &= \left[-\cos\phi+\tfrac{1}{3}\cos^3\phi\right]_0^{\pi}\left[\tfrac{1}{2}\left(\theta-\tfrac{1}{2}\sin 2\theta\right)\right]_0^{\pi}\left[\tfrac{1}{5}\rho^5\right]_0^3\\ &= \left(\tfrac{2}{3}+\tfrac{2}{3}\right)\left(\tfrac{1}{2}\pi\right)\left(\tfrac{1}{5}(243)\right)=\left(\tfrac{4}{3}\right)\left(\tfrac{\pi}{2}\right)\left(\tfrac{243}{5}\right)=\tfrac{162\pi}{5}\end{aligned}$$

25. In spherical coordinates, E is represented by $\left\{(\rho,\theta,\phi)\,\middle|\,0\le\rho\le 1, 0\le\theta\le\frac{\pi}{2}, 0\le\phi\le\frac{\pi}{2}\right\}$. Thus

$$\begin{aligned}\iiint_E xe^{x^2+y^2+z^2}\,dV &= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1(\rho\sin\phi\cos\theta)e^{\rho^2}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi=\int_0^{\pi/2}\sin^2\phi\,d\phi\int_0^{\pi/2}\cos\theta\,d\theta\int_0^1\rho^3e^{\rho^2}\,d\rho\\ &= \int_0^{\pi/2}\tfrac{1}{2}(1-\cos 2\phi)\,d\phi\int_0^{\pi/2}\cos\theta\,d\theta\left(\left[\tfrac{1}{2}\rho^2e^{\rho^2}\right]_0^1-\int_0^1\rho e^{\rho^2}\,d\rho\right)\\ &\qquad\left[\text{integrate by parts with } u=\rho^2,\ dv=\rho e^{\rho^2}\,d\rho\right]\\ &= \left[\tfrac{1}{2}\phi-\tfrac{1}{4}\sin 2\phi\right]_0^{\pi/2}\left[\sin\theta\right]_0^{\pi/2}\left[\tfrac{1}{2}\rho^2e^{\rho^2}-\tfrac{1}{2}e^{\rho^2}\right]_0^1=\left(\tfrac{\pi}{4}-0\right)(1-0)\left(0+\tfrac{1}{2}\right)=\tfrac{\pi}{8}\end{aligned}$$

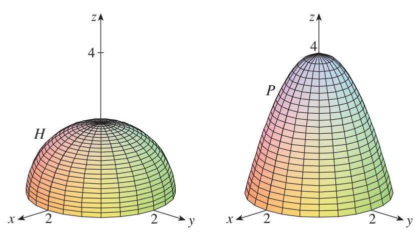
26.
$$\begin{aligned}\iiint_E xyz\,dV &= \int_0^{\pi/3}\int_0^{2\pi}\int_2^4(\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi\\ &= \int_0^{\pi/3}\sin^3\phi\cos\phi\,d\phi\int_0^{2\pi}\sin\theta\cos\theta\,d\theta\int_2^4\rho^5\,d\rho=\left[\tfrac{1}{4}\sin^4\phi\right]_0^{\pi/3}\left[\tfrac{1}{2}\sin^2\theta\right]_0^{2\pi}\left[\tfrac{1}{6}\rho^6\right]_2^4=0\end{aligned}$$

0.0.15 Questions Solutions Stoke Theorem

16.8 Exercises

1. A hemisphere H and a portion P of a paraboloid are shown. Suppose $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ whose components have continuous partial derivatives. Explain why

$$\iint_H \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_P \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$



2–6 Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

2. $\mathbf{F}(x, y, z) = 2y \cos z \, \mathbf{i} + e^x \sin z \, \mathbf{j} + xe^y \, \mathbf{k}$,
S is the hemisphere $x^2 + y^2 + z^2 = 9$, $z \geqslant 0$, oriented upward

3. $\mathbf{F}(x, y, z) = x^2 z^2 \, \mathbf{i} + y^2 z^2 \, \mathbf{j} + xyz \, \mathbf{k}$,
S is the part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 4$, oriented upward

4. $\mathbf{F}(x, y, z) = \tan^{-1}(x^2 yz^2) \, \mathbf{i} + x^2 y \, \mathbf{j} + x^2 z^2 \, \mathbf{k}$,
S is the cone $x = \sqrt{y^2 + z^2}$, $0 \leqslant x \leqslant 2$, oriented in the direction of the positive x-axis

5. $\mathbf{F}(x, y, z) = xyz \, \mathbf{i} + xy \, \mathbf{j} + x^2 yz \, \mathbf{k}$,
S consists of the top and the four sides (but not the bottom) of the cube with vertices $(\pm 1, \pm 1, \pm 1)$, oriented outward

6. $\mathbf{F}(x, y, z) = e^{xy} \, \mathbf{i} + e^{xz} \, \mathbf{j} + x^2 z \, \mathbf{k}$,
S is the half of the ellipsoid $4x^2 + y^2 + 4z^2 = 4$ that lies to the right of the xz-plane, oriented in the direction of the positive y-axis

7–10 Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. In each case C is oriented counterclockwise as viewed from above.

7. $\mathbf{F}(x, y, z) = (x + y^2) \, \mathbf{i} + (y + z^2) \, \mathbf{j} + (z + x^2) \, \mathbf{k}$,
C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$

8. $\mathbf{F}(x, y, z) = \mathbf{i} + (x + yz) \, \mathbf{j} + \left(xy - \sqrt{z}\right) \mathbf{k}$,
C is the boundary of the part of the plane $3x + 2y + z = 1$ in the first octant

9. $\mathbf{F}(x, y, z) = yz \, \mathbf{i} + 2xz \, \mathbf{j} + e^{xy} \, \mathbf{k}$,
C is the circle $x^2 + y^2 = 16$, $z = 5$

10. $\mathbf{F}(x, y, z) = xy \, \mathbf{i} + 2z \, \mathbf{j} + 3y \, \mathbf{k}$, C is the curve of intersection of the plane $x + z = 5$ and the cylinder $x^2 + y^2 = 9$

11. (a) Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y, z) = x^2 z \, \mathbf{i} + xy^2 \, \mathbf{j} + z^2 \, \mathbf{k}$$

and C is the curve of intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 9$ oriented counterclockwise as viewed from above.

(b) Graph both the plane and the cylinder with domains chosen so that you can see the curve C and the surface that you used in part (a).

(c) Find parametric equations for C and use them to graph C.

12. (a) Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = x^2 y \, \mathbf{i} + \frac{1}{3}x^3 \, \mathbf{j} + xy \, \mathbf{k}$ and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$ oriented counterclockwise as viewed from above.

(b) Graph both the hyperbolic paraboloid and the cylinder with domains chosen so that you can see the curve C and the surface that you used in part (a).

(c) Find parametric equations for C and use them to graph C.

13–15 Verify that Stokes' Theorem is true for the given vector field $\mathbf{F}$ and surface S.

13. $\mathbf{F}(x, y, z) = -y \, \mathbf{i} + x \, \mathbf{j} - 2 \, \mathbf{k}$,
S is the cone $z^2 = x^2 + y^2$, $0 \leqslant z \leqslant 4$, oriented downward

14. $\mathbf{F}(x, y, z) = -2yz \, \mathbf{i} + y \, \mathbf{j} + 3x \, \mathbf{k}$,
S is the part of the paraboloid $z = 5 - x^2 - y^2$ that lies above the plane $z = 1$, oriented upward

15. $\mathbf{F}(x, y, z) = y \, \mathbf{i} + z \, \mathbf{j} + x \, \mathbf{k}$,
S is the hemisphere $x^2 + y^2 + z^2 = 1$, $y \geqslant 0$, oriented in the direction of the positive y-axis

16. Let C be a simple closed smooth curve that lies in the plane $x + y + z = 1$. Show that the line integral

$$\int_C z \, dx - 2x \, dy + 3y \, dz$$

depends only on the area of the region enclosed by C and not on the shape of C or its location in the plane.

17. A particle moves along line segments from the origin to the points $(1, 0, 0)$, $(1, 2, 1)$, $(0, 2, 1)$, and back to the origin under the influence of the force field

$$\mathbf{F}(x, y, z) = z^2 \, \mathbf{i} + 2xy \, \mathbf{j} + 4y^2 \, \mathbf{k}$$

Find the work done.

Graphing calculator or computer required **1.** Homework Hints available at stewartcalculus.com

48. $u(x,y,z) = c/\sqrt{x^2+y^2+z^2}$,

$$\mathbf{F} = -K\,\nabla u = -K\left[-\frac{cx}{(x^2+y^2+z^2)^{3/2}}\,\mathbf{i} - \frac{cy}{(x^2+y^2+z^2)^{3/2}}\,\mathbf{j} - \frac{cz}{(x^2+y^2+z^2)^{3/2}}\,\mathbf{k}\right]$$

$$= \frac{cK}{(x^2+y^2+z^2)^{3/2}}\,(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k})$$

and the outward unit normal is $\mathbf{n} = \frac{1}{a}\,(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k})$.

Thus $\mathbf{F}\cdot\mathbf{n} = \frac{cK}{}$ $(x^2+y^2+z^2)$, but on S, $x^2+y^2+z^2=a^2$ so $\mathbf{F}\cdot\mathbf{n} = \frac{cK}{}$. Hence the rate of heat flow

16.8 Stokes' Theorem

2. The boundary curve C is the circle $x^2+y^2=9$, $z=0$ oriented in the counterclockwise direction when viewed from above. A vector equation of C is $\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$, $0 \le t \le 2\pi$, so $\mathbf{r}'(t) = -3\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j}$ and $\mathbf{F}(\mathbf{r}(t)) = 2(3\sin t)(\cos 0)\,\mathbf{i} + e^{3\cos t}(\sin 0)\,\mathbf{j} + (3\cos t)e^{3\sin t}\,\mathbf{k} = 6\sin t\,\mathbf{i} + (3\cos t)e^{3\sin t}\,\mathbf{k}$. Then, by Stokes' Theorem,

$$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^{2\pi}(-18\sin^2 t + 0 + 0)\,dt = -18\left[\tfrac{1}{2}t - \tfrac{1}{4}\sin 2t\right]_0^{2\pi} = -18\pi.$$

3. The paraboloid $z = x^2+y^2$ intersects the cylinder $x^2+y^2=4$ in the circle $x^2+y^2=4$, $z=4$. This boundary curve C should be oriented in the counterclockwise direction when viewed from above, so a vector equation of C is $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 4\,\mathbf{k}$, $0 \le t \le 2\pi$. Then $\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$,

$$\mathbf{F}(\mathbf{r}(t)) = (4\cos^2 t)(16)\,\mathbf{i} + (4\sin^2 t)(16)\,\mathbf{j} + (2\cos t)(2\sin t)(4)\,\mathbf{k} = 64\cos^2 t\,\mathbf{i} + 64\sin^2 t\,\mathbf{j} + 16\sin t\,\cos t\,\mathbf{k},$$

and by Stokes' Theorem,

$$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^{2\pi}(-128\cos^2 t\,\sin t + 128\sin^2 t\,\cos t + 0)\,dt$$
$$= 128\left[\tfrac{1}{3}\cos^3 t + \tfrac{1}{3}\sin^3 t\right]_0^{2\pi} = 0$$

4. The boundary curve C is the circle $y^2+z^2=4$, $x=2$ which should be oriented in the counterclockwise direction when viewed from the front, so a vector equation of C is $\mathbf{r}(t)=2\,\mathbf{i}+2\cos t\,\mathbf{j}+2\sin t\,\mathbf{k}$, $0\le t\le 2\pi$. Then $\mathbf{F}(\mathbf{r}(t))=\tan^{-1}(32\cos t\sin^2 t)\,\mathbf{i}+8\cos t\,\mathbf{j}+16\sin^2 t\,\mathbf{k}$, $\mathbf{r}'(t)=-2\sin t\,\mathbf{j}+2\cos t\,\mathbf{k}$, and $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)=-16\sin t\cos t+32\sin^2 t\cos t$. Thus

$$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^{2\pi}(-16\sin t\cos t+32\sin^2 t\cos t)\,dt$$
$$= \left[-8\sin^2 t + \tfrac{32}{3}\sin^3 t\right]_0^{2\pi} = 0$$

5. C is the square in the plane $z=-1$. Rather than evaluating a line integral around C we can use Equation 3: $\iint_{S_1}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=\oint_C\mathbf{F}\cdot d\mathbf{r}=\iint_{S_2}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$ where S_1 is the original cube without the bottom and S_2 is the bottom face of the cube. $\operatorname{curl}\mathbf{F}=x^2z\,\mathbf{i}+(xy-2xyz)\,\mathbf{j}+(y-xz)\,\mathbf{k}$. For S_2, we choose $\mathbf{n}=\mathbf{k}$ so that C has the same orientation for both surfaces. Then $\operatorname{curl}\mathbf{F}\cdot\mathbf{n}=y-xz=x+y$ on S_2, where $z=-1$. Thus $\iint_{S_2}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=\int_{-1}^{1}\int_{-1}^{1}(x+y)\,dx\,dy=0$ so $\iint_{S_1}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=0$.

6. The boundary curve C is the circle $x^2+z^2=1$, $y=0$ which should be oriented in the counterclockwise direction when viewed from the right, so a vector equation of C is $\mathbf{r}(t)=\cos(-t)\,\mathbf{i}+\sin(-t)\,\mathbf{k}=\cos t\,\mathbf{i}-\sin t\,\mathbf{k}$, $0\le t\le 2\pi$. Then $\mathbf{F}(\mathbf{r}(t))=\mathbf{i}+e^{-\cos t\sin t}\,\mathbf{j}-\cos^2 t\,\sin t\,\mathbf{k}$, $\mathbf{r}'(t)=-\sin t\,\mathbf{i}-\cos t\,\mathbf{k}$, and $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)=-\sin t+\cos^3 t\,\sin t$. Thus

$$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \oint_C\mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^{2\pi}(-\sin t+\cos^3 t\,\sin t)\,dt$$
$$= \left[\cos t - \tfrac{1}{4}\cos^4 t\right]_0^{2\pi} = 0$$

7. $\operatorname{curl}\mathbf{F}=-2z\,\mathbf{i}-2x\,\mathbf{j}-2y\,\mathbf{k}$ and we take the surface S to be the planar region enclosed by C, so S is the portion of the plane $x+y+z=1$ over $D=\{(x,y)\mid 0\le x\le 1, 0\le y\le 1-x\}$. Since C is oriented counterclockwise, we orient S upward. Using Equation 16.7.10, we have $z=g(x,y)=1-x-y$, $P=-2z$, $Q=-2x$, $R=-2y$, and

$$\int_C\mathbf{F}\cdot d\mathbf{r}=\iint_S\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=\iint_D[-(-2z)(-1)-(-2x)(-1)+(-2y)]\,dA$$
$$=\int_0^1\int_0^{1-x}(-2)\,dy\,dx=-2\int_0^1(1-x)\,dx=-1$$

note
Hight is Z = 1 - x - y
r_x X r_y =<-dz/dx, -dz/dy, 1>= <-1, -1, -1>
so we substitute for dS with <-1, -1, 1>

8. $\operatorname{curl}\mathbf{F}=(x-y)\,\mathbf{i}-y\,\mathbf{j}+\mathbf{k}$ and S is the portion of the plane $3x+2y+z=1$ over $D=\left\{(x,y)\mid 0\le x\le\frac{1}{3}, 0\le y\le\frac{1}{2}(1-3x)\right\}$. We orient S upward and use Equation 16.7.10 with $z=g(x,y)=1-3x-2y$: hight

$$\int_C\mathbf{F}\cdot d\mathbf{r}=\iint_S\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=\iint_D[-(x-y)(-3)-(-y)(-2)+1]\,dA=\int_0^{1/3}\int_0^{(1-3x)/2}(1+3x-5y)\,dy\,dx$$
$$=\int_0^{1/3}\left[(1+3x)y-\tfrac{5}{2}y^2\right]_{y=0}^{y=(1-3x)/2}dx=\int_0^{1/3}\left[\tfrac{1}{2}(1+3x)(1-3x)-\tfrac{5}{2}\cdot\tfrac{1}{4}(1-3x)^2\right]dx$$
$$=\int_0^{1/3}\left(-\tfrac{81}{8}x^2+\tfrac{15}{4}x-\tfrac{1}{8}\right)dx=\left[-\tfrac{27}{8}x^3+\tfrac{15}{8}x^2-\tfrac{1}{8}x\right]_0^{1/3}=-\tfrac{1}{8}+\tfrac{5}{24}-\tfrac{1}{24}=\tfrac{1}{24}$$

0.0.16 Questions Solutions Surface area

15.6 Exercises

1–12 Find the area of the surface.

1. The part of the plane $z = 2 + 3x + 4y$ that lies above the rectangle $[0, 5] \times [1, 4]$

2. The part of the plane $2x + 5y + z = 10$ that lies inside the cylinder $x^2 + y^2 = 9$

3. The part of the plane $3x + 2y + z = 6$ that lies in the first octant

4. The part of the surface $z = 1 + 3x + 2y^2$ that lies above the triangle with vertices $(0, 0)$, $(0, 1)$, and $(2, 1)$

5. The part of the cylinder $y^2 + z^2 = 9$ that lies above the rectangle with vertices $(0, 0)$, $(4, 0)$, $(0, 2)$, and $(4, 2)$

6. The part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the xy-plane

7. The part of the hyperbolic paraboloid $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$

8. The surface $z = \frac{2}{3}(x^{3/2} + y^{3/2})$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$

9. The part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$

10. The part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the plane $z = 1$

11. The part of the sphere $x^2 + y^2 + z^2 = a^2$ that lies within the cylinder $x^2 + y^2 = ax$ and above the xy-plane

12. The part of the sphere $x^2 + y^2 + z^2 = 4z$ that lies inside the paraboloid $z = x^2 + y^2$

13–14 Find the area of the surface correct to four decimal places by expressing the area in terms of a single integral and using your calculator to estimate the integral.

13. The part of the surface $z = e^{-x^2-y^2}$ that lies above the disk $x^2 + y^2 \leqslant 4$

14. The part of the surface $z = \cos(x^2 + y^2)$ that lies inside the cylinder $x^2 + y^2 = 1$

15. (a) Use the Midpoint Rule for double integrals (see Section 15.1) with four squares to estimate the surface area of the portion of the paraboloid $z = x^2 + y^2$ that lies above the square $[0, 1] \times [0, 1]$.

CAS (b) Use a computer algebra system to approximate the surface area in part (a) to four decimal places. Compare with the answer to part (a).

16. (a) Use the Midpoint Rule for double integrals with $m = n = 2$ to estimate the area of the surface $z = xy + x^2 + y^2$, $0 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant 2$.

CAS (b) Use a computer algebra system to approximate the surface area in part (a) to four decimal places. Compare with the answer to part (a).

CAS **17.** Find the exact area of the surface $z = 1 + 2x + 3y + 4y^2$, $1 \leqslant x \leqslant 4$, $0 \leqslant y \leqslant 1$.

CAS **18.** Find the exact area of the surface

$$z = 1 + x + y + x^2 \qquad -2 \leqslant x \leqslant 1 \quad -1 \leqslant y \leqslant 1$$

Illustrate by graphing the surface.

CAS **19.** Find, to four decimal places, the area of the part of the surface $z = 1 + x^2y^2$ that lies above the disk $x^2 + y^2 \leqslant 1$.

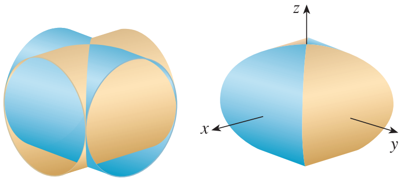
CAS **20.** Find, to four decimal places, the area of the part of the surface $z = (1 + x^2)/(1 + y^2)$ that lies above the square $|x| + |y| \leqslant 1$. Illustrate by graphing this part of the surface.

21. Show that the area of the part of the plane $z = ax + by + c$ that projects onto a region D in the xy-plane with area $A(D)$ is $\sqrt{a^2 + b^2 + 1}\,A(D)$.

22. If you attempt to use Formula 2 to find the area of the top half of the sphere $x^2 + y^2 + z^2 = a^2$, you have a slight problem because the double integral is improper. In fact, the integrand has an infinite discontinuity at every point of the boundary circle $x^2 + y^2 = a^2$. However, the integral can be computed as the limit of the integral over the disk $x^2 + y^2 \leqslant t^2$ as $t \to a^-$. Use this method to show that the area of a sphere of radius a is $4\pi a^2$.

23. Find the area of the finite part of the paraboloid $y = x^2 + z^2$ cut off by the plane $y = 25$. [*Hint:* Project the surface onto the xz-plane.]

24. The figure shows the surface created when the cylinder $y^2 + z^2 = 1$ intersects the cylinder $x^2 + z^2 = 1$. Find the area of this surface.



CAS Computer algebra system required **1.** Homework Hints available at stewartcalculus.com

is again r (see the figure). So

$$\begin{aligned} E &= k \int_{-\pi/2}^{\pi/2} \int_0^{20\cos\theta} \left(1 - \tfrac{1}{20} r\right) r\, dr\, d\theta = k \int_{-\pi/2}^{\pi/2} \left[\tfrac{1}{2} r^2 - \tfrac{1}{60} r^3\right]_{r=0}^{r=20\cos\theta} d\theta \\ &= k \int_{-\pi/2}^{\pi/2} \left(200\cos^2\theta - \tfrac{400}{3}\cos^3\theta\right) d\theta = 200k \int_{-\pi/2}^{\pi/2} \left[\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta - \tfrac{2}{3}\left(1 - \sin^2\theta\right)\cos\theta\right] d\theta \\ &= 200k \left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta - \tfrac{2}{3}\sin\theta + \tfrac{2}{3}\cdot\tfrac{1}{3}\sin^3\theta\right]_{-\pi/2}^{\pi/2} = 200k\left[\tfrac{\pi}{4} + 0 - \tfrac{2}{3} + \tfrac{2}{9} + \tfrac{\pi}{4} + 0 - \tfrac{2}{3} + \tfrac{2}{9}\right] \\ &= 200k\left(\tfrac{\pi}{2} - \tfrac{8}{9}\right) \approx 136k \end{aligned}$$

Therefore the risk of infection is much lower at the edge of the city than in the middle, so it is better to live at the edge.

15.6 Surface Area

1. Here $z = f(x, y) = 2 + 3x + 4y$ and D is the rectangle $[0, 5] \times [1, 4]$, so by Formula 2 the area of the surface is

$$\begin{aligned} A(S) &= \iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\, dA = \iint_D \sqrt{3^2 + 4^2 + 1}\, dA = \sqrt{26} \iint_D dA \\ &= \sqrt{26}\, A(D) = \sqrt{26}\,(5)(3) = 15\sqrt{26} \end{aligned}$$

2. $z = f(x, y) = 10 - 2x - 5y$ and D is the disk $x^2 + y^2 \le 9$, so by Formula 2

$$A(S) = \iint_D \sqrt{(-2)^2 + (-5)^2 + 1}\, dA = \sqrt{30} \iint_D dA = \sqrt{30}\, A(D) = \sqrt{30}\,(\pi \cdot 3^2) = 9\sqrt{30}\,\pi$$

3. $z = f(x, y) = 6 - 3x - 2y$ which intersects the xy-plane in the line $3x + 2y = 6$, so D is the triangular region given by $\left\{(x, y) \mid 0 \le x \le 2, 0 \le y \le 3 - \frac{3}{2}x\right\}$. Thus

$$A(S) = \iint_D \sqrt{(-3)^2 + (-2)^2 + 1}\, dA = \sqrt{14} \iint_D dA = \sqrt{14}\, A(D) = \sqrt{14}\left(\tfrac{1}{2} \cdot 2 \cdot 3\right) = 3\sqrt{14}$$

4. $z = f(x, y) = 1 + 3x + 2y^2$ with $0 \le x \le 2y$, $0 \le y \le 1$. Thus by Formula 2,

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + (3)^2 + (4y)^2}\, dA = \int_0^1 \int_0^{2y} \sqrt{10 + 16y^2}\, dx\, dy = \int_0^1 \sqrt{10 + 16y^2}\,\left[x\right]_{x=0}^{x=2y} dy \\ &= \int_0^1 2y\sqrt{10 + 16y^2}\, dy = 2 \cdot \tfrac{1}{32} \cdot \tfrac{2}{3}\left(10 + 16y^2\right)^{3/2}\Big]_0^1 = \tfrac{1}{24}\left(26^{3/2} - 10^{3/2}\right) \end{aligned}$$

5. $y^2 + z^2 = 9 \quad \Rightarrow \quad z = \sqrt{9 - y^2}$. $f_x = 0$, $f_y = -y(9 - y^2)^{-1/2} \quad \Rightarrow$

$$\begin{aligned} A(S) &= \int_0^4 \int_0^2 \sqrt{0^2 + [-y(9 - y^2)^{-1/2}]^2 + 1}\, dy\, dx = \int_0^4 \int_0^2 \sqrt{\frac{y^2}{9 - y^2} + 1}\, dy\, dx \\ &= \int_0^4 \int_0^2 \frac{3}{\sqrt{9 - y^2}}\, dy\, dx = 3\int_0^4 \left[\sin^{-1}\frac{y}{3}\right]_{y=0}^{y=2} dx = 3\left[\left(\sin^{-1}\left(\tfrac{2}{3}\right)\right)x\right]_0^4 = 12\sin^{-1}\left(\tfrac{2}{3}\right) \end{aligned}$$

6. $z = f(x, y) = 4 - x^2 - y^2$ and D is the projection of the paraboloid $z = 4 - x^2 - y^2$ onto the xy-plane, that is, $D = \left\{(x, y) \mid x^2 + y^2 \le 4\right\}$. So $f_x = -2x$, $f_y = -2y \quad \Rightarrow$

$$\begin{aligned} A(S) &= \iint_D \sqrt{(-2x)^2 + (-2y)^2 + 1}\, dA = \iint_D \sqrt{4(x^2 + y^2) + 1}\, dA = \int_0^{2\pi} \int_0^2 \sqrt{4r^2 + 1}\, r\, dr\, d\theta \\ &= \int_0^{2\pi} \left[\tfrac{1}{12}(4r^2 + 1)^{3/2}\right]_{r=0}^{r=2} d\theta = \int_0^{2\pi} \tfrac{1}{12}\left(17\sqrt{17} - 1\right) d\theta = \tfrac{\pi}{6}\left(17\sqrt{17} - 1\right) \end{aligned}$$

7. $z = f(x,y) = y^2 - x^2$ with $1 \le x^2 + y^2 \le 4$. Then

$$A(S) = \iint_D \sqrt{1+4x^2+4y^2}\,dA = \int_0^{2\pi}\int_1^2 \sqrt{1+4r^2}\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_1^2 r\sqrt{1+4r^2}\,dr$$

$$= \big[\,\theta\,\big]_0^{2\pi}\left[\tfrac{1}{12}(1+4r^2)^{3/2}\right]_1^2 = \tfrac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)$$

8. $z = f(x,y) = \frac{2}{3}(x^{3/2} + y^{3/2})$ and $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1\}$. Then $f_x = x^{1/2}$, $f_y = y^{1/2}$ and

$$A(S) = \iint_D \sqrt{\left(\sqrt{x}\right)^2 + \left(\sqrt{y}\right)^2 + 1}\,dA = \int_0^1\int_0^1 \sqrt{x+y+1}\,dy\,dx = \int_0^1 \left[\tfrac{2}{3}(x+y+1)^{3/2}\right]_{y=0}^{y=1} dx$$

$$= \tfrac{2}{3}\int_0^1 \left[(x+2)^{3/2} - (x+1)^{3/2}\right] dx = \tfrac{2}{3}\left[\tfrac{2}{5}(x+2)^{5/2} - \tfrac{2}{5}(x+1)^{5/2}\right]_0^1$$

$$= \tfrac{4}{15}\left(3^{5/2} - 2^{5/2} - 2^{5/2} + 1\right) = \tfrac{4}{15}\left(3^{5/2} - 2^{7/2} + 1\right)$$

9. $z = f(x,y) = xy$ with $x^2 + y^2 \le 1$, so $f_x = y$, $f_y = x$ $\Rightarrow$

$$A(S) = \iint_D \sqrt{y^2 + x^2 + 1}\,dA = \int_0^{2\pi}\int_0^1 \sqrt{r^2+1}\,r\,dr\,d\theta = \int_0^{2\pi}\left[\tfrac{1}{3}(r^2+1)^{3/2}\right]_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{3}\left(2\sqrt{2} - 1\right) d\theta = \tfrac{2\pi}{3}\left(2\sqrt{2} - 1\right)$$

10. Given the sphere $x^2 + y^2 + z^2 = 4$, when $z = 1$, we get $x^2 + y^2 = 3$ so $D = \left\{(x,y) \mid x^2 + y^2 \le 3\right\}$ and $z = f(x,y) = \sqrt{4 - x^2 - y^2}$. Thus

$$A(S) = \iint_D \sqrt{[(-x)(4-x^2-y^2)^{-1/2}]^2 + [(-y)(4-x^2-y^2)^{-1/2}]^2 + 1}\,dA$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} \sqrt{\frac{r^2}{4-r^2} + 1}\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}} \sqrt{\frac{r^2 + 4 - r^2}{4-r^2}}\,r\,dr\,d\theta$$

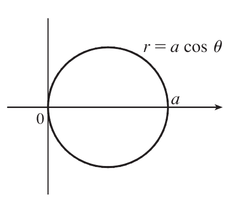
$$= \int_0^{2\pi}\int_0^{\sqrt{3}} \frac{2r}{\sqrt{4-r^2}}\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[-2(4-r^2)^{1/2}\right]_{r=0}^{r=\sqrt{3}} d\theta = \int_0^{2\pi}(-2+4)\,d\theta = 2\theta\big]_0^{2\pi} = 4\pi$$

11. $z = \sqrt{a^2 - x^2 - y^2}$, $z_x = -x(a^2 - x^2 - y^2)^{-1/2}$, $z_y = -y(a^2 - x^2 - y^2)^{-1/2}$,

$$A\,(S) = \iint_D \sqrt{\frac{x^2+y^2}{a^2-x^2-y^2} + 1}\,dA$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta} \sqrt{\frac{r^2}{a^2-r^2} + 1}\,r\,dr\,d\theta$$



$$= \int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta} \frac{ar}{\sqrt{a^2-r^2}}\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left[-a\sqrt{a^2-r^2}\right]_{r=0}^{r=a\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} -a\left(\sqrt{a^2 - a^2\cos^2\theta} - a\right) d\theta = 2a^2\int_0^{\pi/2}\left(1 - \sqrt{1-\cos^2\theta}\right) d\theta$$

$$= 2a^2\int_0^{\pi/2} d\theta - 2a^2\int_0^{\pi/2}\sqrt{\sin^2\theta}\,d\theta = a^2\pi - 2a^2\int_0^{\pi/2}\sin\theta\,d\theta = a^2(\pi - 2)$$

0.0.17 Questions Solutions Surface Area Part II

26. The part of the plane $z = x + 2$ that lies inside the cylinder

33–36 Find an equation of the tangent plane to the given parametric surface at the specified point.

33. $x = u + v, \quad y = 3u^2, \quad z = u - v; \quad (2, 3, 0)$

34. $x = u^2 + 1, \quad y = v^3 + 1, \quad z = u + v; \quad (5, 2, 3)$

35. $\mathbf{r}(u, v) = u \cos v\,\mathbf{i} + u \sin v\,\mathbf{j} + v\,\mathbf{k}; \quad u = 1, \ v = \pi/3$

36. $\mathbf{r}(u, v) = \sin u\,\mathbf{i} + \cos u \sin v\,\mathbf{j} + \sin v\,\mathbf{k};$
$u = \pi/6, \ v = \pi/6$

CAS **37–38** Find an equation of the tangent plane to the given parametric surface at the specified point. Graph the surface and the tangent plane.

37. $\mathbf{r}(u, v) = u^2\,\mathbf{i} + 2u \sin v\,\mathbf{j} + u \cos v\,\mathbf{k}; \quad u = 1, \ v = 0$

38. $\mathbf{r}(u, v) = (1 - u^2 - v^2)\,\mathbf{i} - v\,\mathbf{j} - u\,\mathbf{k}; \quad (-1, -1, -1)$

39–50 Find the area of the surface.

39. The part of the plane $3x + 2y + z = 6$ that lies in the first octant

40. The part of the plane with vector equation $\mathbf{r}(u, v) = \langle u + v, 2 - 3u, 1 + u - v \rangle$ that is given by $0 \leqslant u \leqslant 2, \ -1 \leqslant v \leqslant 1$

41. The part of the plane $x + 2y + 3z = 1$ that lies inside the cylinder $x^2 + y^2 = 3$

42. The part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the plane $y = x$ and the cylinder $y = x^2$

43. The surface $z = \frac{2}{3}(x^{3/2} + y^{3/2}), \ 0 \leqslant x \leqslant 1, \ 0 \leqslant y \leqslant 1$

44. The part of the surface $z = 1 + 3x + 2y^2$ that lies above the triangle with vertices $(0, 0)$, $(0, 1)$, and $(2, 1)$

45. The part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$

46. The part of the paraboloid $x = y^2 + z^2$ that lies inside the cylinder $y^2 + z^2 = 9$

47. The part of the surface $y = 4x + z^2$ that lies between the planes $x = 0$, $x = 1$, $z = 0$, and $z = 1$

CAS

33. $\mathbf{r}(u,v) = (u+v)\,\mathbf{i} + 3u^2\,\mathbf{j} + (u-v)\,\mathbf{k}$.

$\mathbf{r}_u = \mathbf{i} + 6u\,\mathbf{j} + \mathbf{k}$ and $\mathbf{r}_v = \mathbf{i} - \mathbf{k}$, so $\mathbf{r}_u \times \mathbf{r}_v = -6u\,\mathbf{i} + 2\,\mathbf{j} - 6u\,\mathbf{k}$. Since the point $(2,3,0)$ corresponds to $u=1$, $v=1$, a normal vector to the surface at $(2,3,0)$ is $-6\,\mathbf{i} + 2\,\mathbf{j} - 6\,\mathbf{k}$, and an equation of the tangent plane is $-6x + 2y - 6z = -6$ or $3x - y + 3z = 3$.

34. $\mathbf{r}(u,v) = (u^2+1)\,\mathbf{i} + (v^3+1)\,\mathbf{j} + (u+v)\,\mathbf{k}$.

$\mathbf{r}_u = 2u\,\mathbf{i} + \mathbf{k}$ and $\mathbf{r}_v = 3v^2\,\mathbf{j} + \mathbf{k}$, so $\mathbf{r}_u \times \mathbf{r}_v = -3v^2\,\mathbf{i} - 2u\,\mathbf{j} + 6uv^2\,\mathbf{k}$. Since the point $(5,2,3)$ corresponds to $u=2$, $v=1$, a normal vector to the surface at $(5,2,3)$ is $-3\,\mathbf{i} - 4\,\mathbf{j} + 12\,\mathbf{k}$, and an equation of the tangent plane is $-3(x-5) - 4(y-2) + 12(z-3) = 0$ or $3x + 4y - 12z = -13$.

35. $\mathbf{r}(u,v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k} \;\;\Rightarrow\;\; \mathbf{r}\left(1,\frac{\pi}{3}\right) = \left(\frac{1}{2},\frac{\sqrt{3}}{2},\frac{\pi}{3}\right)$.

$\mathbf{r}_u = \cos v\,\mathbf{i} + \sin v\,\mathbf{j}$ and $\mathbf{r}_v = -u\sin v\,\mathbf{i} + u\cos v\,\mathbf{j} + \mathbf{k}$, so a normal vector to the surface at the point $\left(\frac{1}{2},\frac{\sqrt{3}}{2},\frac{\pi}{3}\right)$ is

$\mathbf{r}_u\left(1,\frac{\pi}{3}\right) \times \mathbf{r}_v\left(1,\frac{\pi}{3}\right) = \left(\frac{1}{2}\,\mathbf{i} + \frac{\sqrt{3}}{2}\,\mathbf{j}\right) \times \left(-\frac{\sqrt{3}}{2}\,\mathbf{i} + \frac{1}{2}\,\mathbf{j} + \mathbf{k}\right) = \frac{\sqrt{3}}{2}\,\mathbf{i} - \frac{1}{2}\,\mathbf{j} + \mathbf{k}$. Thus an equation of the tangent plane at

$\left(\frac{1}{2},\frac{\sqrt{3}}{2},\frac{\pi}{3}\right)$ is $\frac{\sqrt{3}}{2}\left(x-\frac{1}{2}\right) - \frac{1}{2}\left(y-\frac{\sqrt{3}}{2}\right) + 1\left(z-\frac{\pi}{3}\right) = 0$ or $\frac{\sqrt{3}}{2}x - \frac{1}{2}y + z = \frac{\pi}{3}$.

36. $\mathbf{r}(u,v) = \sin u\,\mathbf{i} + \cos u\,\sin v\,\mathbf{j} + \sin v\,\mathbf{k} \;\;\Rightarrow\;\; \mathbf{r}\left(\frac{\pi}{6},\frac{\pi}{6}\right) = \left(\frac{1}{2},\frac{\sqrt{3}}{4},\frac{1}{2}\right)$.

$\mathbf{r}_u = \cos u\,\mathbf{i} - \sin u\sin v\,\mathbf{j}$ and $\mathbf{r}_v = \cos u\cos v\,\mathbf{j} + \cos v\,\mathbf{k}$, so a normal vector to the surface at the point $\left(\frac{1}{2},\frac{\sqrt{3}}{4},\frac{1}{2}\right)$ is

$\mathbf{r}_u\left(\frac{\pi}{6},\frac{\pi}{6}\right) \times \mathbf{r}_v\left(\frac{\pi}{6},\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}\,\mathbf{i} - \frac{1}{4}\,\mathbf{j}\right) \times \left(\frac{3}{4}\,\mathbf{j} + \frac{\sqrt{3}}{2}\,\mathbf{k}\right) = -\frac{\sqrt{3}}{8}\,\mathbf{i} - \frac{3}{4}\,\mathbf{j} + \frac{3\sqrt{3}}{8}\,\mathbf{k}$.

Thus an equation of the tangent plane at $\left(\frac{1}{2},\frac{\sqrt{3}}{4},\frac{1}{2}\right)$ is $-\frac{\sqrt{3}}{8}\left(x-\frac{1}{2}\right) - \frac{3}{4}\left(y-\frac{\sqrt{3}}{4}\right) + \frac{3\sqrt{3}}{8}\left(z-\frac{1}{2}\right) = 0$ or

$\sqrt{3}\,x + 6y - 3\sqrt{3}\,z = \frac{\sqrt{3}}{2}$ or $2x + 4\sqrt{3}\,y - 6z = 1$.

37. $\mathbf{r}(u,v) = u^2\,\mathbf{i} + 2u\sin v\,\mathbf{j} + u\cos v\,\mathbf{k} \;\;\Rightarrow\;\; \mathbf{r}(1,0) = (1,0,1)$.

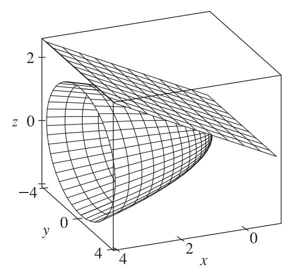
$\mathbf{r}_u = 2u\,\mathbf{i} + 2\sin v\,\mathbf{j} + \cos v\,\mathbf{k}$ and $\mathbf{r}_v = 2u\cos v\,\mathbf{j} - u\sin v\,\mathbf{k}$,

so a normal vector to the surface at the point $(1,0,1)$ is

$\mathbf{r}_u(1,0) \times \mathbf{r}_v(1,0) = (2\,\mathbf{i} + \mathbf{k}) \times (2\,\mathbf{j}) = -2\,\mathbf{i} + 4\,\mathbf{k}$.

Thus an equation of the tangent plane at $(1,0,1)$ is

$-2(x-1) + 0(y-0) + 4(z-1) = 0$ or $-x + 2z = 1$.



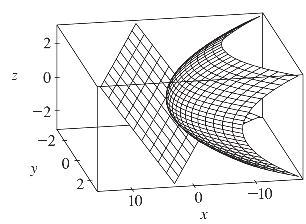
38. $\mathbf{r}(u,v) = (1-u^2-v^2)\,\mathbf{i} - v\,\mathbf{j} - u\,\mathbf{k}$.

$\mathbf{r}_u = -2u\,\mathbf{i} - \mathbf{k}$ and $\mathbf{r}_v = -2v\,\mathbf{i} - \mathbf{j}$. Since the point $(-1,-1,-1)$

corresponds to $u = 1$, $v = 1$, a normal vector to the surface at

$(-1,-1,-1)$ is

$\mathbf{r}_u(1,1) \times \mathbf{r}_v(1,1) = (-2\,\mathbf{i} - \mathbf{k}) \times (-2\,\mathbf{i} - \mathbf{j}) = -\mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k}$.



Thus an equation of the tangent plane is $-1(x+1) + 2(y+1) + 2(z+1) = 0$ or $-x + 2y + 2z = -3$.

39. The surface S is given by $z = f(x,y) = 6 - 3x - 2y$ which intersects the xy-plane in the line $3x + 2y = 6$, so D is the triangular region given by $\left\{(x,y) \mid 0 \le x \le 2, 0 \le y \le 3 - \frac{3}{2}x\right\}$. By Formula 9, the surface area of S is

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA$$
$$= \iint_D \sqrt{1 + (-3)^2 + (-2)^2}\,dA = \sqrt{14}\iint_D dA = \sqrt{14}\,A(D) = \sqrt{14}\left(\tfrac{1}{2}\cdot 2\cdot 3\right) = 3\sqrt{14}.$$

40. $\mathbf{r}_u = \langle 1, -3, 1\rangle$, $\mathbf{r}_v = \langle 1, 0, -1\rangle$, and $\mathbf{r}_u \times \mathbf{r}_v = \langle 3, 2, 3\rangle$. Then by Definition 6,

$$A(S) = \iint_D |\,\mathbf{r}_u \times \mathbf{r}_v\,|\,dA = \int_0^2 \int_{-1}^1 |\,\langle 3, 2, 3\rangle\,|\,dv\,du = \sqrt{22}\int_0^2 du\,\int_{-1}^1 dv = \sqrt{22}\,(2)(2) = 4\sqrt{22}$$

41. Here we can write $z = f(x, y) = \frac{1}{3} - \frac{1}{3}x - \frac{2}{3}y$ and D is the disk $x^2 + y^2 \le 3$, so by Formula 9 the area of the surface is

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA = \iint_D \sqrt{1 + \left(-\tfrac{1}{3}\right)^2 + \left(-\tfrac{2}{3}\right)^2}\,dA = \tfrac{\sqrt{14}}{3}\iint_D dA \\ &= \tfrac{\sqrt{14}}{3}\,A(D) = \tfrac{\sqrt{14}}{3}\cdot \pi\left(\sqrt{3}\right)^2 = \sqrt{14}\,\pi \end{aligned}$$

42. $z = f(x, y) = \sqrt{x^2 + y^2} \quad\Rightarrow\quad \dfrac{\partial z}{\partial x} = \dfrac{1}{2}\left(x^2 + y^2\right)^{-1/2}\cdot 2x = \dfrac{x}{\sqrt{x^2 + y^2}},\ \dfrac{\partial z}{\partial y} = \dfrac{y}{\sqrt{x^2 + y^2}}$, and

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}} = \sqrt{1 + \frac{x^2 + y^2}{x^2 + y^2}} = \sqrt{2}$$

Here D is given by $\left\{(x, y)\;\middle|\; 0 \le x \le 1, x^2 \le y \le x\right\}$, so by Formula 9, the surface area of S is

$$A(S) = \iint_D \sqrt{2}\,dA = \int_0^1 \int_{x^2}^x \sqrt{2}\,dy\,dx = \sqrt{2}\int_0^1 (x - x^2)\,dx = \sqrt{2}\,\left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_0^1 = \sqrt{2}\left(\tfrac{1}{2} - \tfrac{1}{3}\right) = \tfrac{\sqrt{2}}{6}$$

43. $z = f(x, y) = \frac{2}{3}(x^{3/2} + y^{3/2})$ and $D = \{(x, y)\,|\,0 \le x \le 1, 0 \le y \le 1\}$. Then $f_x = x^{1/2}$, $f_y = y^{1/2}$ and

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + \left(\sqrt{x}\,\right)^2 + \left(\sqrt{y}\,\right)^2}\,dA = \int_0^1 \int_0^1 \sqrt{1 + x + y}\,dy\,dx \\ &= \int_0^1 \left[\tfrac{2}{3}(x + y + 1)^{3/2}\right]_{y=0}^{y=1} dx = \tfrac{2}{3}\int_0^1 \left[(x + 2)^{3/2} - (x + 1)^{3/2}\right] dx \\ &= \tfrac{2}{3}\left[\tfrac{2}{5}(x + 2)^{5/2} - \tfrac{2}{5}(x + 1)^{5/2}\right]_0^1 = \tfrac{4}{15}(3^{5/2} - 2^{5/2} - 2^{5/2} + 1) = \tfrac{4}{15}(3^{5/2} - 2^{7/2} + 1) \end{aligned}$$

44. $z = f(x, y) = 1 + 3x + 2y^2$ with $0 \le x \le 2y$, $0 \le y \le 1$. Thus, by Formula 9,

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + 3^2 + (4y)^2}\,dA = \int_0^1 \int_0^{2y} \sqrt{10 + 16y^2}\,dx\,dy = \int_0^1 2y\sqrt{10 + 16y^2}\,dy \\ &= \tfrac{1}{16}\cdot\tfrac{2}{3}(10 + 16y^2)^{3/2}\Big]_0^1 = \tfrac{1}{24}(26^{3/2} - 10^{3/2}) \end{aligned}$$

45. $z = f(x, y) = xy$ with $x^2 + y^2 \le 1$, so $f_x = y$, $f_y = x \quad\Rightarrow$

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + y^2 + x^2}\,dA = \int_0^{2\pi}\int_0^1 \sqrt{r^2 + 1}\,r\,dr\,d\theta = \int_0^{2\pi}\left[\tfrac{1}{3}\left(r^2 + 1\right)^{3/2}\right]_{r=0}^{r=1} d\theta \\ &= \int_0^{2\pi} \tfrac{1}{3}\left(2\sqrt{2} - 1\right) d\theta = \tfrac{2\pi}{3}\left(2\sqrt{2} - 1\right) \end{aligned}$$

46. A parametric representation of the surface is $x = y^2 + z^2$, $y = y$, $z = z$ with $0 \le y^2 + z^2 \le 9$.

Hence $\mathbf{r}_y \times \mathbf{r}_z = (2y\,\mathbf{i} + \mathbf{j}) \times (2z\,\mathbf{i} + \mathbf{k}) = \mathbf{i} - 2y\,\mathbf{j} - 2z\,\mathbf{k}$.

Note: In general, if $x = f(y, z)$ then $\mathbf{r}_y \times \mathbf{r}_z = \mathbf{i} - \dfrac{\partial f}{\partial y}\mathbf{j} - \dfrac{\partial f}{\partial z}\mathbf{k}$, and $A(S) = \displaystyle\iint_D \sqrt{1 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}\,dA$. Then

$$\begin{aligned} A(S) &= \iint\limits_{0 \le y^2 + z^2 \le 9} \sqrt{1 + 4y^2 + 4z^2}\,dA = \int_0^{2\pi}\int_0^3 \sqrt{1 + 4r^2}\,r\,dr\,d\theta \\ &= \int_0^{2\pi} d\theta\,\int_0^3 r\sqrt{1 + 4r^2}\,dr = 2\pi\left[\tfrac{1}{12}(1 + 4r^2)^{3/2}\right]_0^3 = \tfrac{\pi}{6}\left(37\sqrt{37} - 1\right) \end{aligned}$$

0.0.18 Questions Solutions Tangent Line

EXAMPLE 1
(a) Find the derivative of $\mathbf{r}(t) = (1 + t^3)\,\mathbf{i} + te^{-t}\mathbf{j} + \sin 2t\,\mathbf{k}$.
(b) Find the unit tangent vector at the point where $t = 0$.

SOLUTION
(a) According to Theorem 2, we differentiate each component of $\mathbf{r}$:

$$\mathbf{r}'(t) = 3t^2\,\mathbf{i} + (1 - t)e^{-t}\mathbf{j} + 2\cos 2t\,\mathbf{k}$$

(b) Since $\mathbf{r}(0) = \mathbf{i}$ and $\mathbf{r}'(0) = \mathbf{j} + 2\mathbf{k}$, the unit tangent vector at the point $(1, 0, 0)$ is

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{\mathbf{j} + 2\mathbf{k}}{\sqrt{1 + 4}} = \frac{1}{\sqrt{5}}\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k}$$ ■

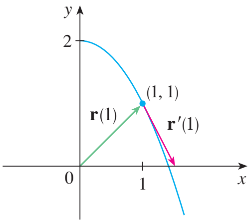


FIGURE 2

Notice from Figure 2 that the tangent vector points in the direction of increasing t. (See Exercise 58.)

EXAMPLE 2 For the curve $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + (2 - t)\,\mathbf{j}$, find $\mathbf{r}'(t)$ and sketch the position vector $\mathbf{r}(1)$ and the tangent vector $\mathbf{r}'(1)$.

SOLUTION We have

$$\mathbf{r}'(t) = \frac{1}{2\sqrt{t}}\mathbf{i} - \mathbf{j} \qquad \text{and} \qquad \mathbf{r}'(1) = \frac{1}{2}\mathbf{i} - \mathbf{j}$$

The curve is a plane curve and elimination of the parameter from the equations $x = \sqrt{t}$, $y = 2 - t$ gives $y = 2 - x^2$, $x \geqslant 0$. In Figure 2 we draw the position vector $\mathbf{r}(1) = \mathbf{i} + \mathbf{j}$ starting at the origin and the tangent vector $\mathbf{r}'(1)$ starting at the corresponding point $(1, 1)$. ■

EXAMPLE 3 Find parametric equations for the tangent line to the helix with parametric equations

$$x = 2\cos t \qquad y = \sin t \qquad z = t$$

at the point $(0, 1, \pi/2)$.

SOLUTION The vector equation of the helix is $\mathbf{r}(t) = \langle 2\cos t, \sin t, t\rangle$, so

$$\mathbf{r}'(t) = \langle -2\sin t, \cos t, 1\rangle$$

The parameter value corresponding to the point $(0, 1, \pi/2)$ is $t = \pi/2$, so the tangent vector there is $\mathbf{r}'(\pi/2) = \langle -2, 0, 1\rangle$. The tangent line is the line through $(0, 1, \pi/2)$ parallel to the vector $\langle -2, 0, 1\rangle$, so by Equations 12.5.2 its parametric equations are

$$x = -2t \qquad y = 1 \qquad z = \frac{\pi}{2} + t$$ ■

The helix and the tangent line in Example 3 are shown in Figure 3.

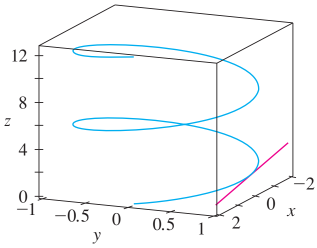


FIGURE 3

13.2 EXERCISES

1. The figure shows a curve C given by a vector function $\mathbf{r}(t)$.

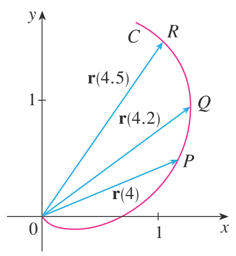
(a) Draw the vectors $\mathbf{r}(4.5) - \mathbf{r}(4)$ and $\mathbf{r}(4.2) - \mathbf{r}(4)$.

(b) Draw the vectors

$$\frac{\mathbf{r}(4.5) - \mathbf{r}(4)}{0.5} \quad \text{and} \quad \frac{\mathbf{r}(4.2) - \mathbf{r}(4)}{0.2}$$

(c) Write expressions for $\mathbf{r}'(4)$ and the unit tangent vector $\mathbf{T}(4)$.

(d) Draw the vector $\mathbf{T}(4)$.



2. (a) Make a large sketch of the curve described by the vector function $\mathbf{r}(t) = \langle t^2, t \rangle$, $0 \leqslant t \leqslant 2$, and draw the vectors $\mathbf{r}(1)$, $\mathbf{r}(1.1)$, and $\mathbf{r}(1.1) - \mathbf{r}(1)$.

(b) Draw the vector $\mathbf{r}'(1)$ starting at $(1, 1)$, and compare it with the vector

$$\frac{\mathbf{r}(1.1) - \mathbf{r}(1)}{0.1}$$

Explain why these vectors are so close to each other in length and direction.

3–8

(a) Sketch the plane curve with the given vector equation.

(b) Find $\mathbf{r}'(t)$.

(c) Sketch the position vector $\mathbf{r}(t)$ and the tangent vector $\mathbf{r}'(t)$ for the given value of t.

3. $\mathbf{r}(t) = \langle t - 2, t^2 + 1 \rangle$, $t = -1$

4. $\mathbf{r}(t) = \langle t^2, t^3 \rangle$, $t = 1$

5. $\mathbf{r}(t) = e^{2t}\,\mathbf{i} + e^t\,\mathbf{j}$, $t = 0$

6. $\mathbf{r}(t) = e^t\,\mathbf{i} + 2t\,\mathbf{j}$, $t = 0$

7. $\mathbf{r}(t) = 4 \sin t\,\mathbf{i} - 2 \cos t\,\mathbf{j}$, $t = 3\pi/4$

8. $\mathbf{r}(t) = (\cos t + 1)\,\mathbf{i} + (\sin t - 1)\,\mathbf{j}$, $t = -\pi/3$

9–16 Find the derivative of the vector function.

9. $\mathbf{r}(t) = \left\langle \sqrt{t - 2}, 3, 1/t^2 \right\rangle$

10. $\mathbf{r}(t) = \langle e^{-t}, t - t^3, \ln t \rangle$

11. $\mathbf{r}(t) = t^2\,\mathbf{i} + \cos(t^2)\,\mathbf{j} + \sin^2 t\,\mathbf{k}$

12. $\mathbf{r}(t) = \dfrac{1}{1 + t}\,\mathbf{i} + \dfrac{t}{1 + t}\,\mathbf{j} + \dfrac{t^2}{1 + t}\,\mathbf{k}$

13. $\mathbf{r}(t) = t \sin t\,\mathbf{i} + e^t \cos t\,\mathbf{j} + \sin t \cos t\,\mathbf{k}$

14. $\mathbf{r}(t) = \sin^2 at\,\mathbf{i} + te^{bt}\,\mathbf{j} + \cos^2 ct\,\mathbf{k}$

15. $\mathbf{r}(t) = \mathbf{a} + t\,\mathbf{b} + t^2\,\mathbf{c}$

16. $\mathbf{r}(t) = t\,\mathbf{a} \times (\mathbf{b} + t\,\mathbf{c})$

17–20 Find the unit tangent vector $\mathbf{T}(t)$ at the point with the given value of the parameter t.

17. $\mathbf{r}(t) = \left\langle t^2 - 2t, 1 + 3t, \frac{1}{3}t^3 + \frac{1}{2}t^2 \right\rangle$, $t = 2$

18. $\mathbf{r}(t) = \langle \tan^{-1} t, 2e^{2t}, 8te^t \rangle$, $t = 0$

19. $\mathbf{r}(t) = \cos t\,\mathbf{i} + 3t\,\mathbf{j} + 2 \sin 2t\,\mathbf{k}$, $t = 0$

20. $\mathbf{r}(t) = \sin^2 t\,\mathbf{i} + \cos^2 t\,\mathbf{j} + \tan^2 t\,\mathbf{k}$, $t = \pi/4$

21. If $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, find $\mathbf{r}'(t)$, $\mathbf{T}(1)$, $\mathbf{r}''(t)$, and $\mathbf{r}'(t) \times \mathbf{r}''(t)$.

22. If $\mathbf{r}(t) = \langle e^{2t}, e^{-2t}, te^{2t} \rangle$, find $\mathbf{T}(0)$, $\mathbf{r}''(0)$, and $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$.

23–26 Find parametric equations for the tangent line to the curve with the given parametric equations at the specified point.

23. $x = t^2 + 1$, $y = 4\sqrt{t}$, $z = e^{t^2 - t}$; $(2, 4, 1)$

24. $x = \ln(t + 1)$, $y = t \cos 2t$, $z = 2^t$; $(0, 0, 1)$

25. $x = e^{-t} \cos t$, $y = e^{-t} \sin t$, $z = e^{-t}$; $(1, 0, 1)$

26. $x = \sqrt{t^2 + 3}$, $y = \ln(t^2 + 3)$, $z = t$; $(2, \ln 4, 1)$

27. Find a vector equation for the tangent line to the curve of intersection of the cylinders $x^2 + y^2 = 25$ and $y^2 + z^2 = 20$ at the point $(3, 4, 2)$.

28. Find the point on the curve $\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, e^t \rangle$, $0 \leqslant t \leqslant \pi$, where the tangent line is parallel to the plane $\sqrt{3}x + y = 1$.

CAS **29–31** Find parametric equations for the tangent line to the curve with the given parametric equations at the specified point. Illustrate by graphing both the curve and the tangent line on a common screen.

29. $x = t$, $y = e^{-t}$, $z = 2t - t^2$; $(0, 1, 0)$

30. $x = 2 \cos t$, $y = 2 \sin t$, $z = 4 \cos 2t$; $\left(\sqrt{3}, 1, 2\right)$

31. $x = t \cos t$, $y = t$, $z = t \sin t$; $(-\pi, \pi, 0)$

32. (a) Find the point of intersection of the tangent lines to the curve $\mathbf{r}(t) = \langle \sin \pi t, 2 \sin \pi t, \cos \pi t \rangle$ at the points where $t = 0$ and $t = 0.5$.

(b) Illustrate by graphing the curve and both tangent lines.

33. The curves $\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle$ and $\mathbf{r}_2(t) = \langle \sin t, \sin 2t, t \rangle$ intersect at the origin. Find their angle of intersection correct to the nearest degree.

21. $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle \Rightarrow \mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$. Then $\mathbf{r}'(1) = \langle 1, 2, 3 \rangle$ and $|\mathbf{r}'(1)| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$, so

23. The vector equation for the curve is $\mathbf{r}(t) = \langle 1 + 2\sqrt{t}, t^3 - t, t^3 + t \rangle$, so $\mathbf{r}'(t) = \langle 1/\sqrt{t}, 3t^2 - 1, 3t^2 + 1 \rangle$. The point $(3, 0, 2)$ corresponds to $t = 1$, so the tangent vector there is $\mathbf{r}'(1) = \langle 1, 2, 4 \rangle$. Thus, the tangent line goes through the point $(3, 0, 2)$ and is parallel to the vector $\langle 1, 2, 4 \rangle$. Parametric equations are $x = 3 + t$, $y = 2t$, $z = 2 + 4t$.

24. The vector equation for the curve is $\mathbf{r}(t) = \left\langle e^t, te^t, te^{t^2} \right\rangle$, so $\mathbf{r}'(t) = \left\langle e^t, te^t + e^t, 2t^2 e^{t^2} + e^{t^2} \right\rangle$. The point $(1, 0, 0)$ corresponds to $t = 0$, so the tangent vector there is $\mathbf{r}'(0) = \langle 1, 1, 1 \rangle$. Thus, the tangent line is parallel to the vector $\langle 1, 1, 1 \rangle$ and includes the point $(1, 0, 0)$. Parametric equations are $x = 1 + 1 \cdot t = 1 + t$, $y = 0 + 1 \cdot t = t$, $z = 0 + 1 \cdot t = t$.

25. The vector equation for the curve is $\mathbf{r}(t) = \langle e^{-t} \cos t, e^{-t} \sin t, e^{-t} \rangle$, so

$$\begin{aligned} \mathbf{r}'(t) &= \langle e^{-t}(-\sin t) + (\cos t)(-e^{-t}), e^{-t} \cos t + (\sin t)(-e^{-t}), (-e^{-t}) \rangle \\ &= \langle -e^{-t}(\cos t + \sin t), e^{-t}(\cos t - \sin t), -e^{-t} \rangle \end{aligned}$$

The point $(1, 0, 1)$ corresponds to $t = 0$, so the tangent vector there is $\mathbf{r}'(0) = \langle -e^0(\cos 0 + \sin 0), e^0(\cos 0 - \sin 0), -e^0 \rangle = \langle -1, 1, -1 \rangle$. Thus, the tangent line is parallel to the vector $\langle -1, 1, -1 \rangle$ and parametric equations are $x = 1 + (-1)t = 1 - t$, $y = 0 + 1 \cdot t = t$, $z = 1 + (-1)t = 1 - t$.

26. The vector equation for the curve is $\mathbf{r}(t) = \langle \sqrt{t^2 + 3}, \ln(t^2 + 3), t \rangle$, so $\mathbf{r}'(t) = \langle t/\sqrt{t^2 + 3}, 2t/(t^2 + 3), 1 \rangle$. At $(2, \ln 4, 1)$, $t = 1$ and $\mathbf{r}'(1) = \langle \frac{1}{2}, \frac{1}{2}, 1 \rangle$. Thus, parametric equations of the tangent line are $x = 2 + \frac{1}{2}t$, $y = \ln 4 + \frac{1}{2}t$, $z = 1 + t$.

27. First we parametrize the curve C of intersection. The projection of C onto the xy-plane is contained in the circle

0.0.19 Questions Solutions Tangent planes

Section 3-1 : Tangent Planes And Linear Approximations - Practice Problems Solutions

2. Find the equation of the tangent plane to $z = x\sqrt{x^2+y^2} + y^3$ at $(-4,3)$.

First, we know we'll need the two 1st order partial derivatives. Here they are,

$$f_x = \sqrt{x^2+y^2} + \frac{x^2}{\sqrt{x^2+y^2}} \qquad f_y = \frac{xy}{\sqrt{x^2+y^2}} + 3y^2$$

Now we also need the two derivatives from the first step and the function evaluated at $(-4,3)$. Here are those evaluations,

$$f(-4,3) = 7 \qquad f_x(-4,3) = \frac{41}{5} \qquad f_y(-4,3) = \frac{123}{5}$$

The tangent plane is then,

$$\boxed{z = 7 + \frac{41}{5}(x+4) + \frac{123}{5}(y-3) = \frac{41}{5}x + \frac{123}{5}y - 34}$$

Section 3-1 : Tangent Planes And Linear Approximations - Practice Problems Solutions

3. Find the linear approximation to $z = 4x^2 - y\mathbf{e}^{2x+y}$ at $(-2, 4)$ .Then approximate f(-1.08, 4.02)

Recall that the linear approximation to a function at a point is really nothing more than the tangent plane to that function at the point.

So, we know that we'll first need the two 1st order partial derivatives. Here they are,

$$f_x = 8x - 2y\mathbf{e}^{2x+y} \qquad\qquad f_y = -\mathbf{e}^{2x+y} - y\mathbf{e}^{2x+y}$$

Now we also need the two derivatives from the first step and the function evaluated at $(-2, 4)$. Here are those evaluations,

$$f(-2,4) = 12 \qquad f_x(-2,4) = -24 \qquad f_y(-2,4) = -5$$

The linear approximation is then,

$$\boxed{L(x,y) = 12 - 24(x+2) - 5(y-4) = -24x - 5y - 16}$$

0.0.20 Questions Solutions Green Theorem

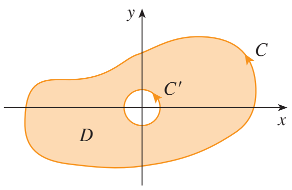


FIGURE 11

with center the origin and radius a, where a is chosen to be small enough that C' lies inside C. (See Figure 11.) Let D be the region bounded by C and C'. Then its positively oriented boundary is $C \cup (-C')$ and so the general version of Green's Theorem gives

$$\int_C P\,dx + Q\,dy + \int_{-C'} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint_D \left[\frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2} \right] dA = 0$$

Therefore

$$\int_C P\,dx + Q\,dy = \int_{C'} P\,dx + Q\,dy$$

that is,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r}$$

We now easily compute this last integral using the parametrization given by $\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j}$, $0 \leqslant t \leqslant 2\pi$. Thus

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$$

$$= \int_0^{2\pi} \frac{(-a\sin t)(-a\sin t) + (a\cos t)(a\cos t)}{a^2\cos^2 t + a^2\sin^2 t}\,dt = \int_0^{2\pi} dt = 2\pi$$

We end this section by using Green's Theorem to discuss a result that was stated in the preceding section.

SKETCH OF PROOF OF THEOREM 16.3.6 We're assuming that $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ is a vector field on an open simply-connected region D, that P and Q have continuous first-order partial derivatives, and that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

If C is any simple closed path in D and R is the region that C encloses, then Green's Theorem gives

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_R 0\,dA = 0$$

A curve that is not simple crosses itself at one or more points and can be broken up into a number of simple curves. We have shown that the line integrals of $\mathbf{F}$ around these simple curves are all 0 and, adding these integrals, we see that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C. Therefore $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D by Theorem 16.3.3. It follows that $\mathbf{F}$ is a conservative vector field.

16.4 Exercises

1–4 Evaluate the line integral by ~~two methods: (a) directly and~~ ~~(b)~~ using Green's Theorem.

1. $\oint_C (x - y)\,dx + (x + y)\,dy$, C is the circle with center the origin and radius 2

2. $\oint_C xy\,dx + x^2\,dy$, C is the rectangle with vertices $(0, 0)$, $(3, 0)$, $(3, 1)$, and $(0, 1)$

3. $\oint_C xy\,dx + x^2y^3\,dy$, C is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 2)$

Graphing calculator or computer required CAS Computer algebra system required 1. Homework Hints available at stewartcalculus.com

4. $\oint_C x^2y^2\,dx + xy\,dy$, C consists of the arc of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$ and the line segments from $(1, 1)$ to $(0, 1)$ and from $(0, 1)$ to $(0, 0)$

5–10 Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

5. $\int_C xy^2\,dx + 2x^2y\,dy$,
C is the triangle with vertices $(0, 0)$, $(2, 2)$, and $(2, 4)$

6. $\int_C \cos y\,dx + x^2 \sin y\,dy$,
C is the rectangle with vertices $(0, 0)$, $(5, 0)$, $(5, 2)$, and $(0, 2)$

7. $\int_C \left(y + e^{\sqrt{x}}\right) dx + (2x + \cos y^2)\,dy$,
C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$

8. $\int_C y^4\,dx + 2xy^3\,dy$, C is the ellipse $x^2 + 2y^2 = 2$

9. $\int_C y^3\,dx - x^3\,dy$, C is the circle $x^2 + y^2 = 4$

10. $\int_C (1 - y^3)\,dx + (x^3 + e^{y^2})\,dy$, C is the boundary of the region between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$

11–14 Use Green's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. (Check the orientation of the curve before applying the theorem.)

11. $\mathbf{F}(x, y) = \langle y \cos x - xy \sin x,\, xy + x \cos x \rangle$,
C is the triangle from $(0, 0)$ to $(0, 4)$ to $(2, 0)$ to $(0, 0)$

12. $\mathbf{F}(x, y) = \langle e^{-x} + y^2,\, e^{-y} + x^2 \rangle$,
C consists of the arc of the curve $y = \cos x$ from $(-\pi/2, 0)$ to $(\pi/2, 0)$ and the line segment from $(\pi/2, 0)$ to $(-\pi/2, 0)$

13. $\mathbf{F}(x, y) = \langle y - \cos y,\, x \sin y \rangle$,
C is the circle $(x - 3)^2 + (y + 4)^2 = 4$ oriented clockwise

14. $\mathbf{F}(x, y) = \left\langle \sqrt{x^2 + 1},\, \tan^{-1} x \right\rangle$, C is the triangle from $(0, 0)$ to $(1, 1)$ to $(0, 1)$ to $(0, 0)$

CAS **15–16** Verify Green's Theorem by using a computer algebra system to evaluate both the line integral and the double integral.

15. $P(x, y) = y^2e^x$, $Q(x, y) = x^2e^y$,
C consists of the line segment from $(-1, 1)$ to $(1, 1)$ followed by the arc of the parabola $y = 2 - x^2$ from $(1, 1)$ to $(-1, 1)$

16. $P(x, y) = 2x - x^3y^5$, $Q(x, y) = x^3y^8$,
C is the ellipse $4x^2 + y^2 = 4$

17. Use Green's Theorem to find the work done by the force $\mathbf{F}(x, y) = x(x + y)\,\mathbf{i} + xy^2\,\mathbf{j}$ in moving a particle from the origin along the x-axis to $(1, 0)$, then along the line segment to $(0, 1)$, and then back to the origin along the y-axis.

18. A particle starts at the point $(-2, 0)$, moves along the x-axis to $(2, 0)$, and then along the semicircle $y = \sqrt{4 - x^2}$ to the starting point. Use Green's Theorem to find the work done on this particle by the force field $\mathbf{F}(x, y) = \langle x, x^3 + 3xy^2 \rangle$.

19. Use one of the formulas in $\boxed{5}$ to find the area under one arch of the cycloid $x = t - \sin t$, $y = 1 - \cos t$.

20. If a circle C with radius 1 rolls along the outside of the circle $x^2 + y^2 = 16$, a fixed point P on C traces out a curve called an *epicycloid*, with parametric equations $x = 5 \cos t - \cos 5t$, $y = 5 \sin t - \sin 5t$. Graph the epicycloid and use $\boxed{5}$ to find the area it encloses.

21. (a) If C is the line segment connecting the point (x_1, y_1) to the point (x_2, y_2), show that

$$\int_C x\,dy - y\,dx = x_1y_2 - x_2y_1$$

(b) If the vertices of a polygon, in counterclockwise order, are $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, show that the area of the polygon is

$$A = \tfrac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)]$$

(c) Find the area of the pentagon with vertices $(0, 0)$, $(2, 1)$, $(1, 3)$, $(0, 2)$, and $(-1, 1)$.

22. Let D be a region bounded by a simple closed path C in the xy-plane. Use Green's Theorem to prove that the coordinates of the centroid $(\bar{x}, \bar{y})$ of D are

$$\bar{x} = \frac{1}{2A} \oint_C x^2\,dy \qquad \bar{y} = -\frac{1}{2A} \oint_C y^2\,dx$$

where A is the area of D.

23. Use Exercise 22 to find the centroid of a quarter-circular region of radius a.

24. Use Exercise 22 to find the centroid of the triangle with vertices $(0, 0)$, $(a, 0)$, and (a, b), where $a > 0$ and $b > 0$.

25. A plane lamina with constant density $\rho(x, y) = \rho$ occupies a region in the xy-plane bounded by a simple closed path C. Show that its moments of inertia about the axes are

$$I_x = -\frac{\rho}{3} \oint_C y^3\,dx \qquad I_y = \frac{\rho}{3} \oint_C x^3\,dy$$

26. Use Exercise 25 to find the moment of inertia of a circular disk of radius a with constant density ρ about a diameter. (Compare with Example 4 in Section 15.5.)

27. Use the method of Example 5 to calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y) = \frac{2xy\,\mathbf{i} + (y^2 - x^2)\,\mathbf{j}}{(x^2 + y^2)^2}$$

and C is any positively oriented simple closed curve that encloses the origin.

28. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle x^2 + y,\, 3x - y^2 \rangle$ and C is the positively oriented boundary curve of a region D that has area 6.

29. If $\mathbf{F}$ is the vector field of Example 5, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every simple closed path that does not pass through or enclose the origin.

36. (a) Here $\mathbf{F}(\mathbf{r}) = c\mathbf{r}/|\mathbf{r}|^3$ and $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Then $f(\mathbf{r}) = -c/|\mathbf{r}|$ is a potential function for $\mathbf{F}$, that is, $\nabla f = \mathbf{F}$. (See the discussion of gradient fields in Section 16.1.) Hence $\mathbf{F}$ is conservative and its line integral is independent of path. Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$.

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = f(P_2) - f(P_1) = -\frac{c}{(x_2^2 + y_2^2 + z_2^2)^{1/2}} + \frac{c}{(x_1^2 + y_1^2 + z_1^2)^{1/2}} = c\left(\frac{1}{d_1} - \frac{1}{d_2}\right).$$

(b) In this case, $c = -(mMG) \quad \Rightarrow$

$$W = -mMG\left(\frac{1}{1.52 \times 10^{11}} - \frac{1}{1.47 \times 10^{11}}\right)$$
$$= -(5.97 \times 10^{24})(1.99 \times 10^{30})(6.67 \times 10^{-11})(-2.2377 \times 10^{-13}) \approx 1.77 \times 10^{32} \text{ J}$$

(c) In this case, $c = \epsilon q Q \quad \Rightarrow$

$$W = \epsilon q Q\left(\frac{1}{10^{-12}} - \frac{1}{5 \times 10^{-13}}\right) = \left(8.985 \times 10^9\right)(1)\left(-1.6 \times 10^{-19}\right)\left(-10^{12}\right) \approx 1400 \text{ J}.$$

16.4 Green's Theorem

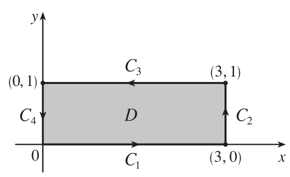
1. (a) Parametric equations for C are $x = 2\cos t,\ \ y = 2\sin t,\ \ 0 \le t \le 2\pi$. Then

$$\oint_C (x-y)\,dx + (x+y)\,dy = \int_0^{2\pi} [(2\cos t - 2\sin t)(-2\sin t) + (2\cos t + 2\sin t)(2\cos t)]\,dt$$
$$= \int_0^{2\pi} (4\sin^2 t + 4\cos^2 t)\,dt = \int_0^{2\pi} 4\,dt = 4t\Big]_0^{2\pi} = 8\pi$$

(b) Note that C as given in part (a) is a positively oriented, smooth, simple closed curve. Then by Green's Theorem,

$$\oint_C (x-y)\,dx + (x+y)\,dy = \iint_D \left[\frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial y}(x-y)\right] dA = \iint_D [1-(-1)]\,dA = 2\iint_D dA$$
$$= 2A(D) = 2\pi(2)^2 = 8\pi$$

2. (a)



C_1: $x = t \ \Rightarrow\ dx = dt,\ \ y = 0 \ \Rightarrow\ dy = 0\,dt,\ \ 0 \le t \le 3.$

C_2: $x = 3 \ \Rightarrow\ dx = 0\,dt,\ \ y = t \ \Rightarrow\ dy = dt,\ \ 0 \le t \le 1.$

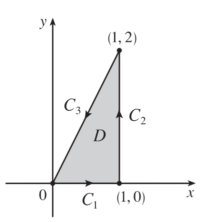
C_3: $x = 3 - t \ \Rightarrow\ dx = -dt,\ \ y = 1 \ \Rightarrow\ dy = 0\,dt,\ \ 0 \le t \le 3.$

C_4: $x = 0 \ \Rightarrow\ dx = 0\,dt,\ \ y = 1 - t \ \Rightarrow\ dy = -dt,\ \ 0 \le t \le 1$

Thus $$\oint_C xy\,dx + x^2\,dy = \oint_{C_1+C_2+C_3+C_4} xy\,dx + x^2\,dy = \int_0^3 0\,dt + \int_0^1 9\,dt + \int_0^3 (3-t)(-1)\,dt + \int_0^1 0\,dt$$
$$= \left[9t\right]_0^1 + \left[\tfrac{1}{2}t^2 - 3t\right]_0^3 = 9 + \tfrac{9}{2} - 9 = \tfrac{9}{2}$$

(b) $\oint_C xy\,dx + x^2\,dy = \iint_D \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy)\right] dA = \int_0^3 \int_0^1 (2x - x)\,dy\,dx = \int_0^3 x\,dx \int_0^1 dy = \left[\tfrac{1}{2}x^2\right]_0^3 \cdot 1 = \tfrac{9}{2}$

3. (a)



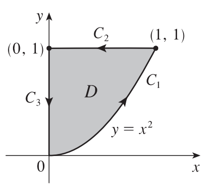
C_1: $x = t \ \Rightarrow\ dx = dt,\ \ y = 0 \ \Rightarrow\ dy = 0\,dt,\ \ 0 \le t \le 1.$

C_2: $x = 1 \ \Rightarrow\ dx = 0\,dt,\ \ y = t \ \Rightarrow\ dy = dt,\ \ 0 \le t \le 2.$

C_3: $x = 1 - t \ \Rightarrow\ dx = -dt,\ \ y = 2 - 2t \ \Rightarrow\ dy = -2\,dt,\ \ 0 \le t \le 1.$

Thus
$$\oint_C xy\,dx + x^2y^3\,dy = \oint_{C_1+C_2+C_3} xy\,dx + x^2y^3\,dy$$
$$= \int_0^1 0\,dt + \int_0^2 t^3\,dt + \int_0^1 \left[-(1-t)(2-2t) - 2(1-t)^2(2-2t)^3\right] dt$$
$$= 0 + \left[\tfrac{1}{4}t^4\right]_0^2 + \left[\tfrac{2}{3}(1-t)^3 + \tfrac{8}{3}(1-t)^6\right]_0^1 = 4 - \tfrac{10}{3} = \tfrac{2}{3}$$

(b) $\oint_C xy\,dx + x^2y^3\,dy = \iint_D \left[\frac{\partial}{\partial x}(x^2y^3) - \frac{\partial}{\partial y}(xy)\right] dA = \int_0^1 \int_0^{2x} (2xy^3 - x)\,dy\,dx$
$$= \int_0^1 \left[\tfrac{1}{2}xy^4 - xy\right]_{y=0}^{y=2x} dx = \int_0^1 (8x^5 - 2x^2)\,dx = \tfrac{4}{3} - \tfrac{2}{3} = \tfrac{2}{3}$$

4. (a) 

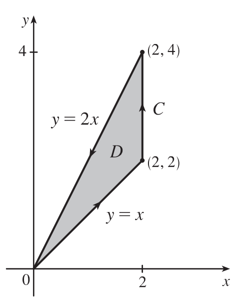
C_1: $x = t \;\Rightarrow\; dx = dt,\; y = t^2 \;\Rightarrow\; dy = 2t\,dt,\; 0 \le t \le 1$

C_2: $x = 1 - t \;\Rightarrow\; dx = -dt,\; y = 1 \;\Rightarrow\; dy = 0\,dt,\; 0 \le t \le 1$

C_3: $x = 0 \;\Rightarrow\; dx = 0\,dt,\; y = 1 - t \;\Rightarrow\; dy = -dt,\; 0 \le t \le 1$

Thus
$$\oint_C x^2y^2\,dx + xy\,dy = \oint_{C_1+C_2+C_3} x^2y^2\,dx + xy\,dy$$
$$= \int_0^1 \left[t^2(t^2)^2\,dt + t(t^2)(2t\,dt)\right] + \int_0^1 \left[(1-t)^2(1)^2(-dt) + (1-t)(1)(0\,dt)\right]$$
$$+ \int_0^1 \left[(0)^2(1-t)^2(0\,dt) + (0)(1-t)(-dt)\right]$$
$$= \int_0^1 \left(t^6 + 2t^4\right) dt + \int_0^1 \left(-1 + 2t - t^2\right) dt + \int_0^1 0\,dt$$
$$= \left[\tfrac{1}{7}t^7 + \tfrac{2}{5}t^5\right]_0^1 + \left[-t + t^2 - \tfrac{1}{3}t^3\right]_0^1 + 0 = \left(\tfrac{1}{7} + \tfrac{2}{5}\right) + \left(-1 + 1 - \tfrac{1}{3}\right) = \tfrac{22}{105}$$

(b) $\oint_C x^2y^2\,dx + xy\,dy = \displaystyle\iint_D \left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x^2y^2)\right] dA = \int_0^1 \int_{x^2}^1 (y - 2x^2y)\,dy\,dx$
$$= \int_0^1 \left[\tfrac{1}{2}y^2 - x^2y^2\right]_{y=x^2}^{y=1} dx = \int_0^1 \left(\tfrac{1}{2} - x^2 - \tfrac{1}{2}x^4 + x^6\right) dx$$
$$= \left[\tfrac{1}{2}x - \tfrac{1}{3}x^3 - \tfrac{1}{10}x^5 + \tfrac{1}{7}x^7\right]_0^1 = \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{10} + \tfrac{1}{7} = \tfrac{22}{105}$$

5. 

The region D enclosed by C is given by $\{(x, y) \mid 0 \le x \le 2, x \le y \le 2x\}$, so
$$\int_C xy^2\,dx + 2x^2y\,dy = \iint_D \left[\frac{\partial}{\partial x}(2x^2y) - \frac{\partial}{\partial y}(xy^2)\right] dA$$
$$= \int_0^2 \int_x^{2x} (4xy - 2xy)\,dy\,dx$$
$$= \int_0^2 \left[xy^2\right]_{y=x}^{y=2x} dx$$
$$= \int_0^2 3x^3\,dx = \left.\tfrac{3}{4}x^4\right]_0^2 = 12$$

6. The region D enclosed by C is $[0, 5] \times [0, 2]$, so
$$\int_C \cos y\,dx + x^2 \sin y\,dy = \iint_D \left[\frac{\partial}{\partial x}(x^2 \sin y) - \frac{\partial}{\partial y}(\cos y)\right] dA = \int_0^5 \int_0^2 \left[2x \sin y - (-\sin y)\right] dy\,dx$$
$$= \int_0^5 (2x + 1)\,dx \int_0^2 \sin y\,dy = \left[x^2 + x\right]_0^5 \left[-\cos y\right]_0^2 = 30(1 - \cos 2)$$

7. $\int_C \left(y+e^{\sqrt{x}}\right)dx+(2x+\cos y^2)\,dy=\iint_D\left[\frac{\partial}{\partial x}(2x+\cos y^2)-\frac{\partial}{\partial y}\left(y+e^{\sqrt{x}}\right)\right]dA$

$$=\int_0^1\int_{y^2}^{\sqrt{y}}(2-1)\,dx\,dy=\int_0^1(y^{1/2}-y^2)\,dy=\tfrac{1}{3}$$

8. $\int_C y^4\,dx+2xy^3\,dy=\iint_D\left[\frac{\partial}{\partial x}(2xy^3)-\frac{\partial}{\partial y}(y^4)\right]dA=\iint_D(2y^3-4y^3)\,dA$

$$=-2\iint_D y^3\,dA=0$$

because $f(x,y)=y^3$ is an odd function with respect to y and D is symmetric about the x-axis.

9. $\int_C y^3\,dx-x^3\,dy=\iint_D\left[\frac{\partial}{\partial x}(-x^3)-\frac{\partial}{\partial y}(y^3)\right]dA=\iint_D(-3x^2-3y^2)\,dA=\int_0^{2\pi}\int_0^2(-3r^2)\,r\,dr\,d\theta$

$$=-3\int_0^{2\pi}d\theta\int_0^2 r^3\,dr=-3(2\pi)(4)=-24\pi$$

10. $\int_C(1-y^3)\,dx+(x^3+e^{y^2})\,dy=\iint_D\left[\frac{\partial}{\partial x}(x^3+e^{y^2})-\frac{\partial}{\partial y}(1-y^3)\right]dA=\iint_D(3x^2+3y^2)\,dA$

$$=\int_0^{2\pi}\int_2^3(3r^2)\,r\,dr\,d\theta=3\int_0^{2\pi}d\theta\int_2^3 r^3\,dr$$

$$=3\big[\theta\big]_0^{2\pi}\left[\tfrac{1}{4}r^4\right]_2^3=3(2\pi)\cdot\tfrac{1}{4}(81-16)=\tfrac{195}{2}\pi$$

11. $\mathbf{F}(x,y)=\langle y\cos x-xy\sin x,\,xy+x\cos x\rangle$ and the region D enclosed by C is given by $\{(x,y)\mid 0\le x\le 2,\,0\le y\le 4-2x\}$. C is traversed clockwise, so $-C$ gives the positive orientation.

$$\int_C\mathbf{F}\cdot d\mathbf{r}=-\int_{-C}(y\cos x-xy\sin x)\,dx+(xy+x\cos x)\,dy=-\iint_D\left[\frac{\partial}{\partial x}(xy+x\cos x)-\frac{\partial}{\partial y}(y\cos x-xy\sin x)\right]dA$$

$$=-\iint_D(y-x\sin x+\cos x-\cos x+x\sin x)\,dA=-\int_0^2\int_0^{4-2x}y\,dy\,dx$$

$$=-\int_0^2\left[\tfrac{1}{2}y^2\right]_{y=0}^{y=4-2x}dx=-\int_0^2\tfrac{1}{2}(4-2x)^2\,dx=-\int_0^2(8-8x+2x^2)\,dx=-\left[8x-4x^2+\tfrac{2}{3}x^3\right]_0^2$$

$$=-\left(16-16+\tfrac{16}{3}-0\right)=-\tfrac{16}{3}$$

12. $\mathbf{F}(x,y)=\left\langle e^{-x}+y^2,\,e^{-y}+x^2\right\rangle$ and the region D enclosed by C is given by $\{(x,y)\mid -\pi/2\le x\le\pi/2,\,0\le y\le\cos x\}$. C is traversed clockwise, so $-C$ gives the positive orientation.

$$\int_C\mathbf{F}\cdot d\mathbf{r}=-\int_{-C}\left(e^{-x}+y^2\right)dx+\left(e^{-y}+x^2\right)dy=-\iint_D\left[\frac{\partial}{\partial x}\left(e^{-y}+x^2\right)-\frac{\partial}{\partial y}\left(e^{-x}+y^2\right)\right]dA$$

$$=-\int_{-\pi/2}^{\pi/2}\int_0^{\cos x}(2x-2y)\,dy\,dx=-\int_{-\pi/2}^{\pi/2}\left[2xy-y^2\right]_{y=0}^{y=\cos x}dx$$

$$=-\int_{-\pi/2}^{\pi/2}(2x\cos x-\cos^2 x)\,dx=-\int_{-\pi/2}^{\pi/2}\left(2x\cos x-\tfrac{1}{2}(1+\cos 2x)\right)dx$$

$$=-\left[2x\sin x+2\cos x-\tfrac{1}{2}\left(x+\tfrac{1}{2}\sin 2x\right)\right]_{-\pi/2}^{\pi/2}\qquad\text{[integrate by parts in the first term]}$$

$$=-\left(\pi-\tfrac{1}{4}\pi-\pi-\tfrac{1}{4}\pi\right)=\tfrac{1}{2}\pi$$

13. $\mathbf{F}(x,y)=\langle y-\cos y,\,x\sin y\rangle$ and the region D enclosed by C is the disk with radius 2 centered at $(3,-4)$. C is traversed clockwise, so $-C$ gives the positive orientation.

$$\int_C\mathbf{F}\cdot d\mathbf{r}=-\int_{-C}(y-\cos y)\,dx+(x\sin y)\,dy=-\iint_D\left[\frac{\partial}{\partial x}(x\sin y)-\frac{\partial}{\partial y}(y-\cos y)\right]dA$$

$$=-\iint_D(\sin y-1-\sin y)\,dA=\iint_D dA=\text{area of }D=\pi(2)^2=4\pi$$

14. $\mathbf{F}(x,y) = \left\langle \sqrt{x^2+1}, \tan^{-1} x \right\rangle$ and the region D enclosed by C is given by $\{(x,y) \mid 0 \le x \le 1, x \le y \le 1\}$.

C is oriented positively, so

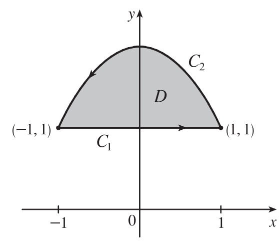
$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \sqrt{x^2+1}\, dx + \tan^{-1} x\, dy = \iint_D \left[\frac{\partial}{\partial x} \left(\tan^{-1} x\right) - \frac{\partial}{\partial y} \left(\sqrt{x^2+1}\right) \right] dA \\ &= \int_0^1 \int_x^1 \left(\frac{1}{1+x^2} - 0 \right) dy\, dx = \int_0^1 \frac{1}{1+x^2} \left[y\right]_{y=x}^{y=1} dx = \int_0^1 \frac{1}{1+x^2} (1-x)\, dx \\ &= \int_0^1 \left(\frac{1}{1+x^2} - \frac{x}{1+x^2} \right) dx = \left[\tan^{-1} x - \frac{1}{2} \ln(1+x^2) \right]_0^1 = \frac{\pi}{4} - \frac{1}{2} \ln 2\end{aligned}$$

15. Here $C = C_1 + C_2$ where

C_1 can be parametrized as $x = t,\ \ y = 1,\ \ -1 \le t \le 1$, and

C_2 is given by $x = -t,\ \ y = 2 - t^2,\ \ -1 \le t \le 1$.

Then the line integral is



$$\begin{aligned}\oint_{C_1+C_2} y^2 e^x\, dx + x^2 e^y\, dy &= \int_{-1}^{1} [1 \cdot e^t + t^2 e \cdot 0]\, dt \\ &\quad + \int_{-1}^{1} [(2-t^2)^2 e^{-t}(-1) + (-t)^2 e^{2-t^2}(-2t)]\, dt \\ &= \int_{-1}^{1} [e^t - (2-t^2)^2 e^{-t} - 2t^3 e^{2-t^2}]\, dt = -8e + 48e^{-1}\end{aligned}$$

according to a CAS. The double integral is

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{-1}^{1} \int_{1}^{2-x^2} (2xe^y - 2ye^x)\, dy\, dx = -8e + 48e^{-1},$$ verifying Green's Theorem in this case.

16. We can parametrize C as $x = \cos\theta,\ \ y = 2\sin\theta,\ \ 0 \le \theta \le 2\pi$. Then the line integral is

$$\begin{aligned}\oint_C P\, dx + Q\, dy &= \int_0^{2\pi} \left[2\cos\theta - (\cos\theta)^3 (2\sin\theta)^5\right] (-\sin\theta)\, d\theta + \int_0^{2\pi} (\cos\theta)^3 (2\sin\theta)^8 \cdot 2\cos\theta\, d\theta \\ &= \int_0^{2\pi} [-2\cos\theta\sin\theta + 32\cos^3\theta\sin^6\theta + 512\cos^4\theta\sin^8\theta]\, d\theta = 7\pi,\end{aligned}$$

according to a CAS. The double integral is $\displaystyle\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{-1}^{1} \int_{-\sqrt{4-4x^2}}^{\sqrt{4-4x^2}} (3x^2 y^8 + 5x^3 y^4)\, dy\, dx = 7\pi.$

17. By Green's Theorem, $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C x(x+y)\, dx + xy^2\, dy = \iint_D (y^2 - x)\, dA$ where C is the path described in the question and D is the triangle bounded by C. So

$$\begin{aligned}W &= \int_0^1 \int_0^{1-x} (y^2 - x)\, dy\, dx = \int_0^1 \left[\tfrac{1}{3} y^3 - xy\right]_{y=0}^{y=1-x} dx = \int_0^1 \left(\tfrac{1}{3}(1-x)^3 - x(1-x)\right) dx \\ &= \left[-\tfrac{1}{12}(1-x)^4 - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3\right]_0^1 = \left(-\tfrac{1}{2} + \tfrac{1}{3}\right) - \left(-\tfrac{1}{12}\right) = -\tfrac{1}{12}\end{aligned}$$

18. By Green's Theorem, $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C x\, dx + (x^3 + 3xy^2)\, dy = \iint_D (3x^2 + 3y^2 - 0)\, dA$, where D is the semicircular region bounded by C. Converting to polar coordinates, we have $W = 3\int_0^2 \int_0^\pi r^2 \cdot r\, d\theta\, dr = 3\pi\left[\tfrac{1}{4}r^4\right]_0^2 = 12\pi$.

19. Let C_1 be the arch of the cycloid from $(0,0)$ to $(2\pi, 0)$, which corresponds to $0 \le t \le 2\pi$, and let C_2 be the segment from $(2\pi, 0)$ to $(0,0)$, so C_2 is given by $x = 2\pi - t$, $y = 0$, $0 \le t \le 2\pi$. Then $C = C_1 \cup C_2$ is traversed clockwise, so $-C$ is

0.0.21 Questions Solutions Divergence

Another application of the Divergence Theorem occurs in fluid flow. Let $\mathbf{v}(x, y, z)$ be the velocity field of a fluid with constant density ρ. Then $\mathbf{F} = \rho\mathbf{v}$ is the rate of flow per unit area. If $P_0(x_0, y_0, z_0)$ is a point in the fluid and B_a is a ball with center P_0 and very small radius a, then $\operatorname{div} \mathbf{F}(P) \approx \operatorname{div} \mathbf{F}(P_0)$ for all points in B_a since $\operatorname{div} \mathbf{F}$ is continuous. We approximate the flux over the boundary sphere S_a as follows:

$$\iint_{S_a} \mathbf{F} \cdot d\mathbf{S} = \iiint_{B_a} \operatorname{div} \mathbf{F}\, dV \approx \iiint_{B_a} \operatorname{div} \mathbf{F}(P_0)\, dV = \operatorname{div} \mathbf{F}(P_0) V(B_a)$$

This approximation becomes better as $a \to 0$ and suggests that

$$\boxed{8} \qquad \operatorname{div} \mathbf{F}(P_0) = \lim_{a \to 0} \frac{1}{V(B_a)} \iint_{S_a} \mathbf{F} \cdot d\mathbf{S}$$

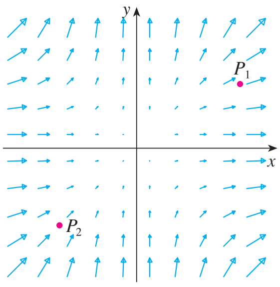


FIGURE 4
The vector field $\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j}$

Equation 8 says that $\operatorname{div} \mathbf{F}(P_0)$ is the net rate of outward flux per unit volume at P_0. (This is the reason for the name *divergence*.) If $\operatorname{div} \mathbf{F}(P) > 0$, the net flow is outward near P and P is called a **source**. If $\operatorname{div} \mathbf{F}(P) < 0$, the net flow is inward near P and P is called a **sink**.

For the vector field in Figure 4, it appears that the vectors that end near P_1 are shorter than the vectors that start near P_1. Thus the net flow is outward near P_1, so $\operatorname{div} \mathbf{F}(P_1) > 0$ and P_1 is a source. Near P_2, on the other hand, the incoming arrows are longer than the outgoing arrows. Here the net flow is inward, so $\operatorname{div} \mathbf{F}(P_2) < 0$ and P_2 is a sink. We can use the formula for $\mathbf{F}$ to confirm this impression. Since $\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j}$, we have $\operatorname{div} \mathbf{F} = 2x + 2y$, which is positive when $y > -x$. So the points above the line $y = -x$ are sources and those below are sinks.

16.9 Exercises

1–4 Verify that the Divergence Theorem is true for the vector field $\mathbf{F}$ on the region E.

1. $\mathbf{F}(x, y, z) = 3x\,\mathbf{i} + xy\,\mathbf{j} + 2xz\,\mathbf{k}$,
E is the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 1$

2. $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + xy\,\mathbf{j} + z\,\mathbf{k}$,
E is the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane

3. $\mathbf{F}(x, y, z) = \langle z, y, x\rangle$,
E is the solid ball $x^2 + y^2 + z^2 \leqslant 16$

4. $\mathbf{F}(x, y, z) = \langle x^2, -y, z\rangle$,
E is the solid cylinder $y^2 + z^2 \leqslant 9$, $0 \leqslant x \leqslant 2$

5–15 Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$; that is, calculate the flux of $\mathbf{F}$ across S.

5. $\mathbf{F}(x, y, z) = xye^z\,\mathbf{i} + xy^2z^3\,\mathbf{j} - ye^z\,\mathbf{k}$,
S is the surface of the box bounded by the coordinate planes and the planes $x = 3$, $y = 2$, and $z = 1$

6. $\mathbf{F}(x, y, z) = x^2yz\,\mathbf{i} + xy^2z\,\mathbf{j} + xyz^2\,\mathbf{k}$,
S is the surface of the box enclosed by the planes $x = 0$, $x = a$, $y = 0$, $y = b$, $z = 0$, and $z = c$, where a, b, and c are positive numbers

7. $\mathbf{F}(x, y, z) = 3xy^2\,\mathbf{i} + xe^z\,\mathbf{j} + z^3\,\mathbf{k}$,
S is the surface of the solid bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = -1$ and $x = 2$

8. $\mathbf{F}(x, y, z) = (x^3 + y^3)\,\mathbf{i} + (y^3 + z^3)\,\mathbf{j} + (z^3 + x^3)\,\mathbf{k}$,
S is the sphere with center the origin and radius 2

9. $\mathbf{F}(x, y, z) = x^2\sin y\,\mathbf{i} + x\cos y\,\mathbf{j} - xz\sin y\,\mathbf{k}$,
S is the "fat sphere" $x^8 + y^8 + z^8 = 8$

10. $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + zx\,\mathbf{k}$,
S is the surface of the tetrahedron enclosed by the coordinate planes and the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

where a, b, and c are positive numbers

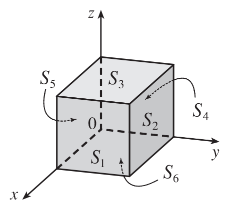
11. $\mathbf{F}(x, y, z) = (\cos z + xy^2)\,\mathbf{i} + xe^{-z}\,\mathbf{j} + (\sin y + x^2z)\,\mathbf{k}$,
S is the surface of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$

12. $\mathbf{F}(x, y, z) = x^4\,\mathbf{i} - x^3z^2\,\mathbf{j} + 4xy^2z\,\mathbf{k}$,
S is the surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = x + 2$ and $z = 0$

13. $\mathbf{F} = |\mathbf{r}|\,\mathbf{r}$, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$,
S consists of the hemisphere $z = \sqrt{1 - x^2 - y^2}$ and the disk $x^2 + y^2 \leqslant 1$ in the xy-plane

CAS Computer algebra system required **1.** Homework Hints available at stewartcalculus.com

16.9 The Divergence Theorem



1. $\operatorname{div}\mathbf{F} = 3 + x + 2x = 3 + 3x$, so

$\iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^1\int_0^1\int_0^1 (3x+3)\,dx\,dy\,dz = \frac{9}{2}$ (notice the triple integral is three times the volume of the cube plus three times $\overline{x}$).

To compute $\iint_S \mathbf{F}\cdot d\mathbf{S}$, on

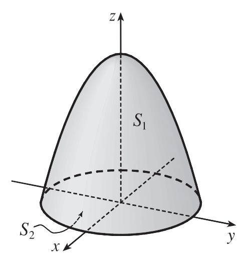
S_1: $\mathbf{n} = \mathbf{i}$, $\mathbf{F} = 3\,\mathbf{i} + y\,\mathbf{j} + 2z\,\mathbf{k}$, and $\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} 3\,dS = 3$;

S_2: $\mathbf{F} = 3x\,\mathbf{i} + x\,\mathbf{j} + 2xz\,\mathbf{k}$, $\mathbf{n} = \mathbf{j}$ and $\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_2} x\,dS = \frac{1}{2}$;

S_3: $\mathbf{F} = 3x\,\mathbf{i} + xy\,\mathbf{j} + 2x\,\mathbf{k}$, $\mathbf{n} = \mathbf{k}$ and $\iint_{S_3}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_3} 2x\,dS = 1$;

S_4: $\mathbf{F} = \mathbf{0}$, $\iint_{S_4}\mathbf{F}\cdot d\mathbf{S} = 0$; S_5: $\mathbf{F} = 3x\,\mathbf{i} + 2x\,\mathbf{k}$, $\mathbf{n} = -\mathbf{j}$ and $\iint_{S_5}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_5} 0\,dS = 0$;

S_6: $\mathbf{F} = 3x\,\mathbf{i} + xy\,\mathbf{j}$, $\mathbf{n} = -\mathbf{k}$ and $\iint_{S_6}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_6} 0\,dS = 0$. Thus $\iint_S \mathbf{F}\cdot d\mathbf{S} = \frac{9}{2}$.



2. $\operatorname{div}\mathbf{F} = 2x + x + 1 = 3x + 1$ so

$$\begin{aligned}
\iiint_E \operatorname{div}\mathbf{F}\,dV &= \iiint_E (3x+1)\,dV = \int_0^{2\pi}\int_0^2\int_0^{4-r^2} (3r\cos\theta + 1)\,r\,dz\,dr\,d\theta \\
&= \int_0^2\int_0^{2\pi} r(3r\cos\theta+1)(4-r^2)\,d\theta\,dr \\
&= \int_0^{2\pi} r(4-r^2)\big[3r\sin\theta + \theta\big]_{\theta=0}^{\theta=2\pi}\,dr \\
&= 2\pi\int_0^2 (4r - r^3)\,dr = 2\pi\big[2r^2 - \tfrac{1}{4}r^4\big]_0^2 \\
&= 2\pi(8-4) = 8\pi
\end{aligned}$$

On S_1: The surface is $z = 4 - x^2 - y^2$, $x^2 + y^2 \le 4$, with upward orientation, and $\mathbf{F} = x^2\,\mathbf{i} + xy\,\mathbf{j} + (4 - x^2 - y^2)\,\mathbf{k}$. Then

$$\begin{aligned}
\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} &= \iint_D [-(x^2)(-2x) - (xy)(-2y) + (4 - x^2 - y^2)]\,dA \\
&= \iint_D \left[2x(x^2+y^2) + 4 - (x^2+y^2)\right]\,dA = \int_0^{2\pi}\int_0^2 (2r\cos\theta\cdot r^2 + 4 - r^2)\,r\,dr\,d\theta \\
&= \int_0^{2\pi}\left[\tfrac{2}{5}r^5\cos\theta + 2r^2 - \tfrac{1}{4}r^4\right]_{r=0}^{r=2}\,d\theta = \int_0^{2\pi}\left(\tfrac{64}{5}\cos\theta + 4\right)\,d\theta = \left[\tfrac{64}{5}\sin\theta + 4\theta\right]_0^{2\pi} = 8\pi
\end{aligned}$$

On S_2: The surface is $z = 0$ with downward orientation, so $\mathbf{F} = x^2\,\mathbf{i} + xy\,\mathbf{j}$, $\mathbf{n} = -\mathbf{k}$ and $\iint_{S_2}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_2} 0\,dS = 0$.

Thus $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1}\mathbf{F}\cdot d\mathbf{S} + \iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = 8\pi$.

3. $\operatorname{div}\mathbf{F} = 0 + 1 + 0 = 1$, so $\iiint_E \operatorname{div}\mathbf{F}\,dV = \iiint_E 1\,dV = V(E) = \frac{4}{3}\pi\cdot 4^3 = \frac{256}{3}\pi$. S is a sphere of radius 4 centered at the origin which can be parametrized by $\mathbf{r}(\phi,\theta) = \langle 4\sin\phi\cos\theta, 4\sin\phi\sin\theta, 4\cos\phi\rangle$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$ (similar to Example 16.6.10). Then

$$\begin{aligned}
\mathbf{r}_\phi \times \mathbf{r}_\theta &= \langle 4\cos\phi\cos\theta, 4\cos\phi\sin\theta, -4\sin\phi\rangle \times \langle -4\sin\phi\sin\theta, 4\sin\phi\cos\theta, 0\rangle \\
&= \left\langle 16\sin^2\phi\cos\theta, 16\sin^2\phi\sin\theta, 16\cos\phi\sin\phi\right\rangle
\end{aligned}$$

and $\mathbf{F}(\mathbf{r}(\phi,\theta)) = \langle 4\cos\phi, 4\sin\phi\sin\theta, 4\sin\phi\cos\theta\rangle$. Thus

$\mathbf{F}\cdot(\mathbf{r}_\phi \times \mathbf{r}_\theta) = 64\cos\phi\sin^2\phi\cos\theta + 64\sin^3\phi\sin^2\theta + 64\cos\phi\sin^2\phi\cos\theta = 128\cos\phi\sin^2\phi\cos\theta + 64\sin^3\phi\sin^2\theta$

and

$$\begin{aligned}
\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D \mathbf{F}\cdot(\mathbf{r}_\phi \times \mathbf{r}_\theta)\,dA = \int_0^{2\pi}\int_0^{\pi} (128\cos\phi\sin^2\phi\cos\theta + 64\sin^3\phi\sin^2\theta)\,d\phi\,d\theta \\
&= \int_0^{2\pi}\left[\tfrac{128}{3}\sin^3\phi\cos\theta + 64\left(-\tfrac{1}{3}(2+\sin^2\phi)\cos\phi\right)\sin^2\theta\right]_{\phi=0}^{\phi=\pi}\,d\theta \\
&= \int_0^{2\pi}\tfrac{256}{3}\sin^2\theta\,d\theta = \tfrac{256}{3}\left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} = \tfrac{256}{3}\pi
\end{aligned}$$

4. $\operatorname{div}\mathbf{F} = 2x - 1 + 1 = 2x$, so

$$\iiint_E \operatorname{div}\mathbf{F}\,dV = \iint_{y^2+z^2\le 9}\left[\int_0^2 2x\,dx\right]dA = \iint_{y^2+z^2\le 9} 4\,dA = 4(\text{area of circle}) = 4(\pi\cdot 3^2) = 36\pi$$

Let S_1 be the front of the cylinder (in the plane $x=2$), S_2 the back (in the yz-plane), and S_3 the lateral surface of the cylinder.

S_1 is the disk $x=2$, $y^2+z^2\le 9$. A unit normal vector is $\mathbf{n} = \langle 1,0,0\rangle$ and $\mathbf{F} = \langle 4,-y,z\rangle$ on S_1, so

$\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_1} 4\,dS = 4(\text{surface area of } S_1) = 4(\pi\cdot 3^2) = 36\pi$. S_2 is the disk $x=0$, $y^2+z^2\le 9$.

Here $\mathbf{n} = \langle -1,0,0\rangle$ and $\mathbf{F} = \langle 0,-y,z\rangle$, so $\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_2}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_2} 0\,dS = 0$.

S_3 can be parametrized by $\mathbf{r}(x,\theta) = \langle x, 3\cos\theta, 3\sin\theta\rangle$, $0\le x\le 2$, $0\le\theta\le 2\pi$. Then

$\mathbf{r}_x\times\mathbf{r}_\theta = \langle 1,0,0\rangle\times\langle 0,-3\sin\theta,3\cos\theta\rangle = \langle 0,-3\cos\theta,-3\sin\theta\rangle$. For the outward (positive) orientation we use

$-(\mathbf{r}_x\times\mathbf{r}_\theta)$ and $\mathbf{F}(\mathbf{r}(x,\theta)) = \langle x^2,-3\cos\theta,3\sin\theta\rangle$, so

$$\begin{aligned}\iint_{S_3}\mathbf{F}\cdot d\mathbf{S} &= \iint_D \mathbf{F}\cdot(-(\mathbf{r}_x\times\mathbf{r}_\theta))\,dA = \int_0^2\int_0^{2\pi}(0-9\cos^2\theta+9\sin^2\theta)\,d\theta\,dx\\ &= -9\int_0^2 dx\int_0^{2\pi}\cos 2\theta\,d\theta = -9\,(2)\left[\tfrac12\sin 2\theta\right]_0^{2\pi} = 0\end{aligned}$$

Thus $\iint_S \mathbf{F}\cdot d\mathbf{S} = 36\pi + 0 + 0 = 36\pi$.

5. $\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(xye^z) + \frac{\partial}{\partial y}(xy^2z^3) + \frac{\partial}{\partial z}(-ye^z) = ye^z + 2xyz^3 - ye^z = 2xyz^3$, so by the Divergence Theorem,

$$\begin{aligned}\iint_S\mathbf{F}\cdot d\mathbf{S} &= \iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^3\int_0^2\int_0^1 2xyz^3\,dz\,dy\,dx = 2\int_0^3 x\,dx\int_0^2 y\,dy\int_0^1 z^3\,dz\\ &= 2\left[\tfrac12x^2\right]_0^3\left[\tfrac12y^2\right]_0^2\left[\tfrac14z^4\right]_0^1 = 2\left(\tfrac92\right)(2)\left(\tfrac14\right) = \tfrac92\end{aligned}$$

6. $\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(xy^2z) + \frac{\partial}{\partial z}(xyz^2) = 2xyz + 2xyz + 2xyz = 6xyz$, so by the Divergence Theorem,

$$\begin{aligned}\iint_S\mathbf{F}\cdot d\mathbf{S} &= \iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^a\int_0^b\int_0^c 6xyz\,dz\,dy\,dx = 6\int_0^a x\,dx\int_0^b y\,dy\int_0^c z\,dz\\ &= 6\left[\tfrac12x^2\right]_0^a\left[\tfrac12y^2\right]_0^b\left[\tfrac12z^2\right]_0^c = 6\left(\tfrac12a^2\right)\left(\tfrac12b^2\right)\left(\tfrac12c^2\right) = \tfrac34a^2b^2c^2\end{aligned}$$

7. $\operatorname{div}\mathbf{F} = 3y^2 + 0 + 3z^2$, so using cylindrical coordinates with $y = r\cos\theta$, $z = r\sin\theta$, $x = x$ we have

$$\begin{aligned}\iint_S\mathbf{F}\cdot d\mathbf{S} &= \iiint_E(3y^2+3z^2)\,dV = \int_0^{2\pi}\int_0^1\int_{-1}^2(3r^2\cos^2\theta + 3r^2\sin^2\theta)\,r\,dx\,dr\,d\theta\\ &= 3\int_0^{2\pi}d\theta\int_0^1 r^3\,dr\int_{-1}^2 dx = 3(2\pi)\left(\tfrac14\right)(3) = \tfrac{9\pi}{2}\end{aligned}$$

8. $\operatorname{div}\mathbf{F} = 3x^2 + 3y^2 + 3z^2$, so by the Divergence Theorem,

$$\begin{aligned}\iint_S\mathbf{F}\cdot d\mathbf{S} &= \iiint_E 3(x^2+y^2+z^2)\,dV = \int_0^\pi\int_0^{2\pi}\int_0^2 3\rho^2\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = 3\int_0^\pi\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_0^2\rho^4\,d\rho\\ &= 3\left[-\cos\phi\right]_0^\pi\left[\theta\right]_0^{2\pi}\left[\tfrac15\rho^5\right]_0^2 = 3\,(2)\,(2\pi)\left(\tfrac{32}{5}\right) = \tfrac{384}{5}\pi\end{aligned}$$

9. $\operatorname{div}\mathbf{F} = 2x\sin y - x\sin y - x\sin y = 0$, so by the Divergence Theorem, $\iint_S\mathbf{F}\cdot d\mathbf{S} = \iiint_E 0\,dV = 0$.

0.1 Worked out Solutions for all Assessment Tools

0.1.1 Solution for Quiz I

Name Meriam Mkadmi, ID 83776

Quiz I , MTH 203 , Fall 2021

Ayman Badawi

QUESTION 1. Convince me that $L_1 : x = 3t + 1, y = t - 2, z = -4t + 2$ ($t \in R$) is perpendicular to $L_2 : x = 2w, y = 2w - 5, z = 2w - 6$ ($w \in R$).

① First condition: $D_1 \bullet D_2 = 0$.

$D_1 = \langle 3, 1, -4 \rangle$ and $D_2 = \langle 2, 2, 2 \rangle$

$$D_1 \bullet D_2 = \langle 3,1,-4 \rangle \bullet \langle 2,2,2 \rangle$$
$$= 3(2) + 1(2) - 4(2) = 0 \checkmark$$

② Second condition: L_1 and L_2 intersect

Check if x in L_1 = x in L_2
y in L_1 = y in L_2
z in L_1 = z in L_2.

x: $3t + 1 = 2w \longrightarrow t = 1$
y: $t - 2 = 2w - 5 \longrightarrow w = 2$

z: $-4t + 2 \stackrel{?}{=} 2w - 6$
$-2 = -2$ $\checkmark$

equation for z is satisfied by t and w.

$$\boxed{\text{Both conditions are met, therefore, } L_1 \perp L_2.}$$

QUESTION 2. Is the line $L_1 : x = 3t + 1, y = t - 2, z = -4t + 2$ ($t \in R$) parallel to $L_2 : x = 6w - 5, y = 2w - 4, z = -8w + 11$ ($w \in R$)? EXPLAIN WHY YES or WHY NO.

1 First condition: $D_1 = cD_2$

$D_1 = \langle 3, 1, -4 \rangle$ and $D_2 = \langle 6, 2, -8 \rangle$

$$\left.\begin{aligned} 3 = 6c &\rightarrow c = 1/2 \\ 1 = 2c &\rightarrow c = 1/2 \\ -4 = -8c &\rightarrow c = 1/2 \end{aligned}\right\}$$

$c = 1/2$ satisfies all of them. ✓

2 Second condition: L_1 does not intersect L_2.

Let's see if the point $(1, -2, 2)$, which lies on L_1, does not lie on L_2.

$$L_2: \left.\begin{aligned} 1 = 6w - 5 &\rightarrow w = 1 \\ -2 = 2w - 4 &\rightarrow w = 1 \\ 2 = -8w + 11 &\rightarrow w = 9/8 \end{aligned}\right\}$$

not satisfied by the same w.

Since the point $(1, -2, 2)$ does not lie on L_2, we can confirm that they are not the same line.

Both conditions are met, therefore $L_1 \parallel L_2$.

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0.1.2 Solution for Quiz II

Quiz II , MTH 203 , Fall 2021

Ayman Badawi

QUESTION 1. Does the line $L : x = -t + 2, y = 2t + 6, z = 3t - 2 (t \in R)$ intersect the plane $P : x + y + 2z = 25$. If yes, then find the point of intersection.

QUESTION 2. Can we draw the vector $v =< 2, -4, -3 >$ inside the plane $P : x + y - 2z = 12$? explain

QUESTION 3. Does the plane $P_1 : x + 2y + z = 5$ intersect the plane $P_2 : -x - y + 3z = -2$? If yes, the find the parametric equations of line of intersection.

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0.1.3 Solution for Quiz III

Name————————————————, ID ————————

Quiz III , MTH 203 , Fall 2021

Ayman Badawi

QUESTION 1. Find all critical points of the function $f(x, y) = yx^2 - 2x^2 - y^2$. Then classify each critical point as local Max/Min or Saddle or neither.

Solution:
F_x = 2xy - 4x = 0, 2x(y - 2) = 0, y = 2 or x = 0 (1)

F_y = x^2 - 2y = 0. y = x^2/2. (2)

assume x = 0, then by (Eq (2), we have y = 0, (0, 0)

In eq(2) we set y = 2, solve for x . 2 = x^2/2, 4 = x^2, x = 2, -2. Hence we get (2, 2), (-2, 2)

Critical points: (0, 0), (2, 2) and (-2, 2)

f_xx = 2y - 4, f_yy = -2 , f_xy = 2x

(0, 0): fxx(0, 0) = -4, fyy(0,0) = -2, fxy(0, 0) = 0.
D = fxx(0, 0)fyy(0,0) - fxy(0, 0)^2 = 8, f_xx(0, 0) <0, we have local max at (0, 0)

(2, 2): fxx(2, 2) = 0, fyy(2, 2) = -2, fxy(2, 2) = 4. D = -16 <0 , saddle point at (2, 2)
(-2, 2): fxx(-2, 2) = 0, fyy(-2, 2) = -2, fxy(-2,2) = -4. D = -16 <0, saddle point at (-2, 2)

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0.1.4 Solution for Quiz IV

Meriam Mkadmi 83776

QUESTION 1. Let $f(x,y) = ye^{(x-4)} + x\sqrt{y} + 2x$.

(i) Let $a = f(4,1)$. Find a

(ii) Find f_x and f_y.

(iii) Let P be the tangent plane to $f(x,y)$ at the point $(4,1,a)$. Let N be a vector that is perpendicular to P. Find N.

(iv) Find the equation of P, where P is as in (iii).

(v) Use the concept of the tangent plane to approximate $f(4.2, 0.8)$

i) $f(4,1) = e^0 + 4 + 2(4) = 13$ ✓ 2

ii) $f_x = ye^{x-4} + \sqrt{y} + 2$ ✓ 2

4/4

$f_y = e^{x-4} + \frac{1}{2}xy^{-\frac{1}{2}} = e^{x-4} + \frac{x}{2\sqrt{y}}$ ✓ 2

iii) $N = \langle f_x(4,1), f_y(4,1), -1 \rangle$

$N = \langle 4, 3, -1 \rangle$ ✓ 2

iv) $M = \overrightarrow{P_1A} = \langle x-4, y-1, z-13 \rangle$

$N \cdot M = 0$

Equation of P:

$4(x-4) + 3(y-1) - (z-13) = 0$ ✓ 2

v) Rearrange to make z the subject

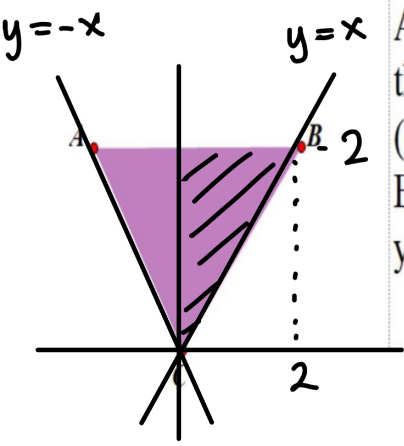
$z = 4(x-4) + 3(y-1) + 13$

$L(x,y) = 4(x-4) + 3(y-1) + 13$

$L(4.2, 0.8) = 4(4.2-4) + 3(0.8-1) + 13$

$= 13.2 \approx f(4.2, 0.8).$

QUESTION 2. A solid object has a a triangular base that is bounded by $y = x$ and $y = -x$ (see PICTURE). Note that $-2 \leq x \leq 2$ and $0 \leq y \leq 2$. The height is determined by the function $f(x, y) = e^{\sqrt{4-y^2}}$. Find the volume of such object.



A solid object has a triangular base in the xy-plane as in picture. Note A = (-2, 2), B = (2, 2), and C = (0, 0). C, B lie on the line y = x, C and A lie on y = -x.

$$0 \leq y \leq 2$$
$$0 \leq x \leq y$$

$$V = 2\left[\int_{y=0}^{y=2}\int_{x=0}^{x=y} e^{\sqrt{4-y^2}}\,dx\,dy\right]$$

First evaluate inner integral:

$$\int_{x=0}^{x=y} e^{\sqrt{4-y^2}}\,dx = xe^{\sqrt{4-y^2}}\Big|_{x=0}^{x=y} = ye^{\sqrt{4-y^2}}$$

Then evaluate outer integral:

$$\int_{y=0}^{y=2} ye^{\sqrt{4-y^2}}\,dy$$

use substitution

$u = 4-y^2 \quad du = -2y\,dy$

$\Rightarrow dy = -\dfrac{du}{2y}$

$$\int_{u=4}^{u=0} e^{\sqrt{u}} \cdot -\frac{du}{2\cancel{y}}\cancel{y} = -\frac{1}{2}\int_{u=4}^{u=0} e^{\sqrt{u}}\,du$$

$= \cancel{2.3890561}$

not easy to integrate

Don't forget to multiply by 2.

$V = \cancel{8.3890561} \times 2 = \cancel{16.778}$

$V = \cancel{16.778}$ units^3

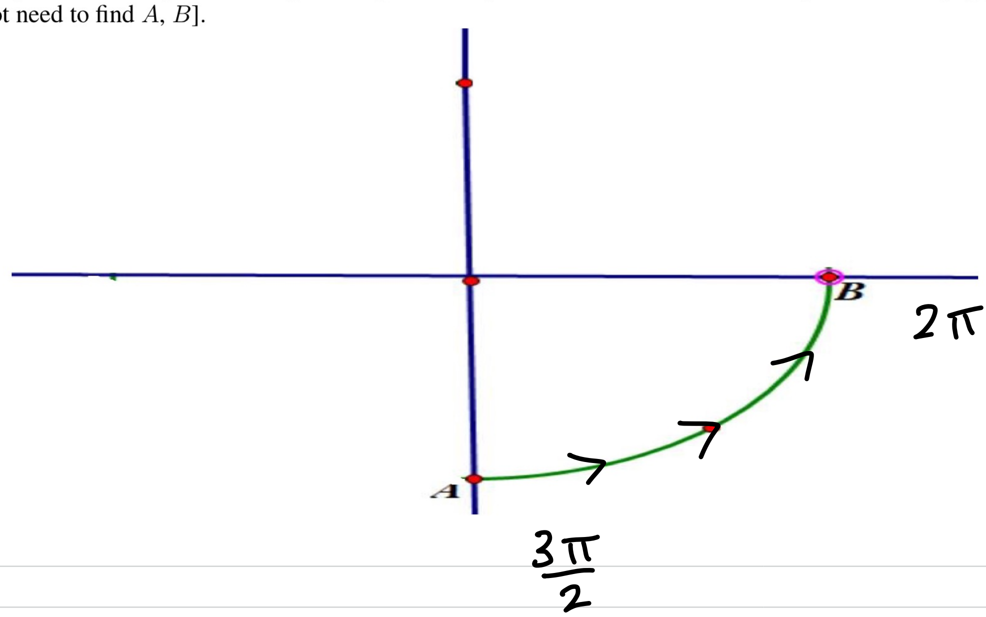
correct

My Fault

0.1.5 Solution for Quiz V

Meriam Mkadmi 83776

QUESTION 1. See the below picture. A force $F(x, y) =< -y, x >$ is acting on a particle in order to move it from the point A to the point B along the ellipse $x^2 + 4y^2 = 16$. Find the work done by the force $F(x, y)$. [Hint: you do not need to find A, B].



Parametric Equations:

$$\left.\begin{aligned} x &= 4\cos(t) \\ y &= 2\sin(t) \end{aligned}\right\} \quad \frac{3\pi}{2} \leq t \leq 2\pi$$

$$dx = -4\sin t\, dt$$

$$dy = 2\cos t\, dt$$

$$dr = \langle dx, dy \rangle$$

$$= \langle -4\sin t\, dt, 2\cos t\, dt \rangle$$

$$\text{Work} = \int_{\frac{3\pi}{2}}^{2\pi} \langle -2\sin t, 4\cos t \rangle \cdot \langle -4\sin t\, dt, 2\cos t\, dt \rangle$$

$$= \int_{\frac{3\pi}{2}}^{2\pi} (8\sin^2 t\, dt + 8\cos^2 t\, dt)$$

$$= \int_{\frac{3\pi}{2}}^{2\pi} 8\,dt = 8t \Big|_{\frac{3\pi}{2}}^{2\pi} = [16\pi - 12\pi]$$

$$= \underline{\underline{4\pi}}$$

QUESTION 2. The height of a curtain is determine by $f(x, y) = e^{2x} + 5y^2$ defined over the curve $y = e^x$, $0 \leq x \leq ln(5)$. Find the surface area of the curtain.

Surface Area = Height × Arc Length

$$\frac{dy}{dx} = e^x$$

$$S.A. = \int_0^{\ln 5} (e^{2x} + 5e^{2x}) \sqrt{1 + e^{2x}}\, dx$$

$$= \int_0^{\ln 5} 6e^{2x} \sqrt{1 + e^{2x}}\, dx$$

Use substitution: $u = 1 + e^{2x}$ $du = 2e^{2x}\,dx$

$\hookrightarrow dx = \frac{1}{2e^{2x}}\,du$

$$= \int_2^{26} 6\cancel{e^{2x}} \sqrt{u} \cdot \frac{1}{2\cancel{e^{2x}}} \, du$$

$$= \int_2^{26} 3\sqrt{u} \, du = 2u^{3/2} \Big|_2^{26}$$

$$= 2(26)^{3/2} - 2(2)^{3/2} = \underline{\underline{259.492\ldots}}$$

QUESTION 3. Evaluate the integral $\int_C (1 + 2y)\,dy$, where the curve C is $y = e^{x^2}$, $0 \le x \le 1$

Rewrite everything in terms of x.

$$y = e^{x^2} \longrightarrow dy = 2xe^{x^2} dx$$

$$\int_0^1 (1+2e^{x^2})\, 2xe^{x^2} dx$$

Use substitution: $u = 1 + 2e^{x^2}$

$du = 4xe^{x^2} dx$

$\to dx = \frac{1}{4xe^{x^2}} du$

$$\int_{3}^{1+2e} (u)\, 2\cancel{xe^{x^2}} \cdot \frac{1}{4\cancel{xe^{x^2}}}\, du$$

$$= \int_{3}^{1+2e} \frac{1}{2} u\, du = \frac{1}{4} u^2 \Big|_{3}^{1+2e}$$

$$= \frac{1}{4}(1+2e)^2 - \frac{1}{4}(3)^2 = \underline{\underline{8.1073\ldots}}$$

0.1.6 Solution for Quiz VI

Quiz VI, MTH 203 , Fall 2021

Ayman Badawi

SHOW THE WORK, SUBMIT by 2:35pm

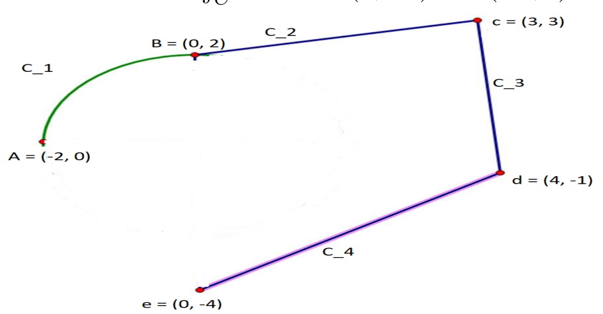
QUESTION 1. See the below picture. A force $F(x,y) =< ye^{yx} - 2x, xe^{yx} + 4y^3 >$ is acting on a particle in order to move it from the point $A = (-2, 0)$ to the point $e = (0, -4)$ along the curve C that consists of C_1 part of the circle $x^2 + y^2 = 4$, C_2 the line segment between B and c, C_3 the line segment between c and d, and C_4 the line segment between d and e. Find the work done by the force $F(x, y)$.

Solution: $f_x = ye^{xy} - 2x$**, hence** $f_{xy} = e^{xy} + xye^{xy}$**.** $f_y = xe^{xy} + 4y^3$**, thus** $f_{yx} = e^{xy} + xye^{xy}$**. Hence** $f_{xy} = f_{yx}$ **and therefore** F **is conservative. So, the line integral does not depend on the path. So the answer is** $K(terminal\ point) - K(initial\ point)$**, where** $K_x = f_x$ **and** $K_y = f_y$**. To find** $K(x, y)$**. We do the following (as in class):**

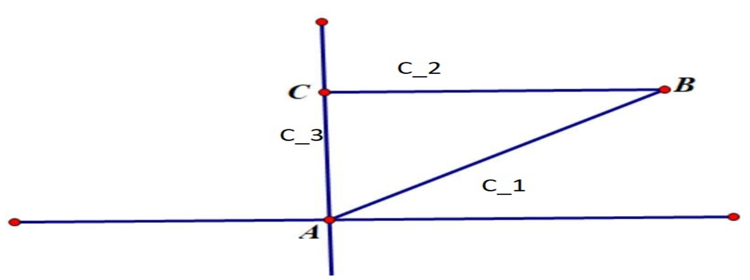
$$\int f_x\, dx = \int ye^{xy} - 2x\, dx = e^{xy} - x^2$$

$$\int f_y\, dy = \int xe^{xy} + 4y^3\, dy = e^{xy} + y^4$$

Now K(x, y) = all terms of $\int fx\, dx$ **+ the terms in** $\int f_y\, dy$ **that are MISSING in** $\int f_x\, dx$
Hence $K(x, y) = e^{xy} - x^2 + y^4$**.**
Thus the work = $\int_C F.dr = K(0, -4) - K(-2, 0) = e^0 + 0 + 256 - (e^0 - 4 + 0) = 260$



QUESTION 2. See the below picture. A force $F(x,y) =< -2y,\ \frac{2}{3}x\sqrt{y^2+9} >$ is acting on a particle in order to move it from the point $A = (0.0)$ then back to the same point A along the curve C that consists of C_1 part of the the line $y = 0.5x$ between A and B, C_2 part of the line $y = 4$ between B and C, and C_3 part of the y-axis between C and A. Use Green's Theorem to find the work done by the force $F(x, y)$. [Hint: Is dxdy or dydx easier?]



Solution: By staring, $\int\int - - -\, dxdy$ **is easier. Ok, we use Green's Theorem:**

$f_x = -2y$**, so** $f_{xy} = -2$**.** $f_y = \frac{2}{3}x\sqrt{y^2+9}$**, so** $f_{yx} = \frac{2}{3}\sqrt{y^2+9}$**. Since** C **is closed simple arc, Green'theorem says** $\int_C F.dr = \int_{C_1} F.dr + \int_{C_2} F.dr + \int_{C_3} F.dr = \int_{y=0}^{y=4}\int_{x=0}^{x=2y} f_{yx} - f_{xy}\, dxdy$**.**

$$\int_{y=0}^{y=4}\int_{x=0}^{x=2y} \frac{2}{3}\sqrt{y^2+9} + 2dxdy$$

.

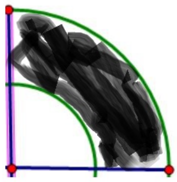
$\int_{x=0}^{x=2y} \frac{2}{3}\sqrt{y^2+9} + 2\, dx = \int_{x=0}^{x=2y} \frac{2}{3}\sqrt{y^2+9}\, dx + \int_{x=0}^{x=2y} 2dx = \frac{2}{3}x\sqrt{y^2+9} + 2x|_{x=0}^{x=2y} = \frac{4y}{3}\sqrt{y^2+9} + 4y$
Now $\int_{y=0}^{y=4} \frac{4y}{3}\sqrt{y^2+9} + 4y\, dy = \int_{y=0}^{y=4} \frac{4y}{3}\sqrt{y^2+9} + \int_{y=0}^{y=4} 4y\, dy$
For $\int_{y=0}^{y=4} \frac{4y}{3}\sqrt{y^2+9}\, dy$ **we use substitution**

Let $u = y^2 + 9$**. Then** $du = 2ydu$ **and** u **is between 9 and 25. Hence we have** $\int_{u=9}^{u=25} \frac{2}{3}u^{0.5}\ du = \frac{4}{9}u^{\frac{3}{2}}|_{u=9}^{u=25} = \frac{4}{9}(125-27) = \frac{392}{9}$

Also, $\int_{y=0}^{y=4} 4y\ dy = 2y^2|_{y=0}^{y=4} = 32$**.**

Thus the answer is $\frac{392}{9} + 32 = \frac{392+288}{9} = \frac{680}{9}$

QUESTION 3. Find the surface area of the part of $f(x,y) = \frac{x^2}{2\sqrt{2}} + \frac{y^2}{2\sqrt{2}}$ defined over the region in the first quadrant of the xy-plane bounded by $x^2 + y^2 \leq 4$ and $x^2 + y^2 \geq 1$, see picture.



Solution: The region is between the two circles (as in the picture), so POLAR is recommended. By staring at the region, we realize that $1 \leq r \leq 2$ **and** $0 \leq \theta \leq \pi/2$ **Let** $f(x,y) = z = \frac{x^2}{2\sqrt{2}} + \frac{y^2}{2\sqrt{2}}$ **.**

Surface Area = $\int_{\theta=0}^{\theta=\pi/2} \int_{r=1}^{r=2} \sqrt{1 + f_x^2 + f_y^2}\ r dr d\theta$**.**

Now we must write $\sqrt{1 + f_x^2 + f_y^2}$ **in terms of** r **and** θ **(since the integration is in terms of** $drd\theta$**)**

Thus $\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + 0.5x^2 + 0.5y^2} = \sqrt{1 + 0.5(x^2+y^2)} = \sqrt{1 + 0.5r^2}$ **(note that every point** (x,y) **between the two circles satisfies** $x^2 + y^2 = r^2$**, where** $1 \leq r \leq 2$**)**

Thus the surface area = $\int_{\theta=0}^{\theta=\pi/2} \int_{r=1}^{r=2} \sqrt{1 + 0.5r^2}\ r dr d\theta$**.**

For $\int_{r=1}^{r=2} \sqrt{1 + 0.5r^2}\ r dr d$ **we use substitution. Let** $u = 1 + 0.5r^2$**. Hence** $du = rdr$ **and** u **is between** 1.5 **and** 3**. Thus** $\int_{u=1.5}^{u=3} u^{0.5}\ du = \frac{2}{3}u^{\frac{3}{2}}|_{u=1.5}^{u=3} = \frac{2}{3}(3^{1.5} - 1.5^{1.5})$

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0.1.7 Solution for Exam I

Midterm 1

Meriam Mkadmi
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Question 1:

$$\text{Height} = \frac{36y}{9x^4+1} = \frac{36x^3}{9x^4+1}$$

$$\frac{dy}{dx} = 3x^2$$

$$S.A. = \int_0^1 \frac{36x^3}{9x^4+1} \cdot \sqrt{1+(3x^2)^2}\, dx$$

$$= \int_0^1 \frac{36x^3\sqrt{1+9x^4}}{9x^4+1}\, dx$$

$$= \int_0^1 \frac{36x^3}{\sqrt{9x^4+1}}\, dx$$

$u = 9x^4+1 \qquad du = 36x^3\, dx$

$dx = \frac{1}{36x^3}\, du$

$x=0 \Rightarrow u=1$

$x=1 \Rightarrow u=10$

$$= \int_{1}^{10} \frac{1}{\sqrt{u}}\, du = \int_{1}^{10} u^{-\frac{1}{2}}\, du$$

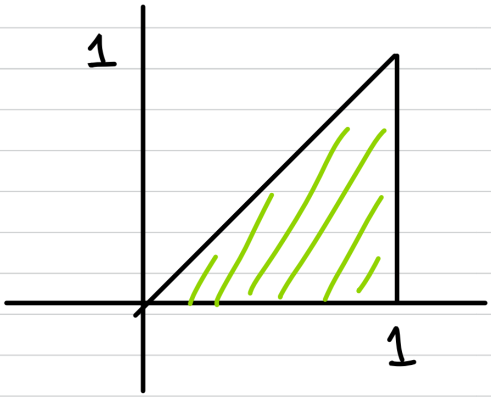
$$= 2\sqrt{u}\,\Big|_{1}^{10} = 2\sqrt{10} - 2$$

$$= \cancel{2\sqrt{10}\ \text{units}^2}$$

or 4.32 units^2

5/6

Question 2:



$0 \leq x \leq 1$
$0 \leq y \leq x$

$$V = \int_0^1 \int_0^x \left(x\sqrt{y+1}\right) dy\, dx$$

First evaluate inner integral (dy)

$$\int_0^x \left(x\sqrt{y+1}\right) dy = \int_0^x \left(x(y+1)^{1/2}\right) dy$$

$$= \left.\frac{x(y+1)^{3/2}}{3/2}\right|_0^x = \left[\frac{2x(x+1)^{3/2}}{3}\right] - \left[\frac{2x}{3}\right]$$

$$= \frac{2x(x+1)^{3/2} - 2x}{3}$$

Now evaluate outer interval (dx)

$$\int_0^1 \left(\frac{2x(x+1)^{3/2} - 2x}{3} \right) dx$$

$$= \int_0^1 \frac{2x(x+1)^{3/2}}{3} dx - \int_0^1 \frac{2x}{3} dx$$

$$= \int_0^1 \frac{2x(x+1)^{3/2}}{3} dx - \left[\frac{x^2}{3} \right]_0^1$$

$$= \int_0^1 \frac{2x(x+1)^{3/2}}{3} dx - \frac{1}{3}$$

$$= \underline{0.3893\ldots \text{ units}^3}$$

Calcul.

ok

6/6

ii) $f_{xx} = -2$ $\quad f_{yy} = 2y$ $\quad f_{xy} = 2$

For $(3.5, 1)$:

$$D = f_{xx}(3.5,1) \cdot f_{yy}(3.5,1) - \left[f_{xy}(3.5,1)\right]^2$$

$$= -2 \cdot 2(1) - 2^2 = -8$$

Since $D < 0$, $(3.5, 1)$ is a saddle point.

For $(-0.5, -3)$:

$$D = f_{xx}(-0.5,-3) \cdot f_{yy}(-0.5,-3) - \left[f_{xy}(-0.5,-3)\right]^2$$

$$= -2 \cdot 2(-3) - 2^2 = 8$$

Since $D > 0$, and $f_{xx} = -2 < 0$, the point $(-0.5, -3)$ is a local maximum with a value of $\underbrace{15.25}$.

$$f(-0.5,-3) = 5(-0.5) - 8(-3) + 2(-0.5)(-3) - (-0.5)^2 + \frac{1}{3}(-3)^2$$

Question 4:

First find the critical point.

$f_x = 2x + 12 = 0 \longrightarrow x = -6$

$f_y = -2y + 6 = 0 \longrightarrow y = 3$

Critical point is $(-6, 3)$, however, $(-6)^2 + (3)^2 \geqslant 25$ so it is not inside the region.

Now we check the circle.

$$x^2 + y^2 = 25$$
$$y^2 = 25 - x^2$$
$$y = \sqrt{25 - x^2}$$

$$f(x,y) = x^2 + 12x - y^2 + 6y$$
$$= x^2 + 12x - (25 - x^2) + 6\sqrt{25 - x^2}$$
$$= x^2 + 12x - 25 + x^2 + 6\sqrt{25 - x^2}$$
$$= 2x^2 + 12x - 25 + 6\sqrt{25 - x^2}$$

8/8

Question 5:

$N_1 = \langle 2, 4, 5 \rangle \qquad N_2 = \langle 6, a, b \rangle$

If $P_1 \parallel P_2$, then $N_1 \parallel N_2$.

$$N_1 = m N_2$$

$2 = 6m \longrightarrow m = 1/3$

$4 = ma \longrightarrow 4 = \frac{1}{3}a \longrightarrow \boxed{a = 12}$

$5 = mb \longrightarrow 5 = \frac{1}{3}b \longrightarrow \boxed{b = 15}$

Choose a point on P_1, let's say $(1,1,0)$.
If $P_1 \parallel P_2$ then this point cannot lie on P_2. So:

$$6(1) + 12(1) + 15(0) \neq c$$

$$\boxed{c \neq 18}$$

6/6

Good ✓

Question 6:

$N_1 \times N_2$ will be the directional vector D of the line L.

$N_1 = \langle 4, 11, -5 \rangle$ $N_2 = \langle -2, -5, -1 \rangle$

$$N_1 \times N_2 = \begin{vmatrix} 4 & 11 & -5 \\ -2 & -5 & -1 \end{vmatrix} = \langle -36, 14, 2 \rangle = D$$

We need a point that lies on P_1 and P_2.

Let $z = 0$.

P_1: $4x + 11y = -3$

$+$

P_2: $(-2x - 5y = 3) \times 2$

$$y = 3$$

$$x = \frac{-3 - 11(3)}{4} = -9$$

The Point is $(-9, 3, 0)$.

$$L: \langle -36, 14, 2 \rangle t + (-9, 3, 0)$$

$$L: \left. \begin{aligned} x &= -36t - 9 \\ y &= 14t + 3 \\ z &= 2t \end{aligned} \right\} t \in \mathbb{R}$$

Question 7:

i) We can rearrange the equation into this form:

$$4x^2 + 9y^2 + z^2 = 25$$

$$\frac{4}{25}x^2 + \frac{9}{25}y^2 + \frac{1}{25}z^2 = 1$$

$$\frac{1}{25/4}x^2 + \frac{1}{25/9}y^2 + \frac{1}{25}z^2 = 1$$

From this form we can see the equation resembles the general equation of an ellipsoid.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

where $a = 25/4$, $b = 25/9$, $c = 25$.

It is an ellipsoid because the coefficients of x^2, y^2, and z^2 differ, so it won't be a perfect sphere.

2

ii) Neither. $x^2+y^2=9$ is a cylinder of infinite height and with radius 3. This is because z is not bounded within a certain interval $a \leq z \leq b$ so the equation will result in infinite circles of radius 3 stacked on top of each other with every changing x and y.

iii) Local max means it is a critical point.

So $f_x(3,-5)=0$ and $f_y(3,-5)=0$.

$N = \langle 0, 0, -1 \rangle$.

$M = \overrightarrow{P_1A} = \langle x-3, y+5, 6-z \rangle$

$N \cdot M = 0$

Tangent Plane:

$-(6-z) = 0 \longrightarrow z = 6$.

2

iv)

1) $f(1,4) = 3(1)(4) + (1)^2 - \sqrt{4} + (1)^3(4)$

$a = 15$

2) $f_x = 3y + 2x + 3x^2 y$

$f_y = 3x - \frac{1}{2} y^{-\frac{1}{2}} + x^3$

$f_x(1,4) = 26$ $f_y(1,4) = 3.75$

$N = \langle f_x(1,4), f_y(1,4), -1 \rangle$

$N = \langle 26, 3.75, -1 \rangle$

$M = \overrightarrow{P_1 A} = \langle x-1, y-4, z-15 \rangle$

$N \cdot M = 0$

Tangent Plane:

$26(x-1) + 3.75(y-4) - (z-15) = 0$

3) Rearrange to make z the subject.

$$z = L(x,y) = 26(x-1) + 3.75(y-4) + 15$$

$$L(0.8, 4.01) = 26(0.8-1) + 3.75(4.01-4) + 15$$
$$= 9.8375 \approx f(1,4)$$

8

Question 8:

$$\frac{dy}{dx} = \frac{-f_x}{f_y}$$

$$f_x = 5y^2e^{5x} + \sin(3y) + 2x + y^3$$

$$f_y = 2ye^{5x} + 3x\cos(3y) + 3xy^2$$

$$\frac{dy}{dx} = \frac{-(5y^2e^{5x} + \sin(3y) + 2x + y^3)}{2ye^{5x} + 3x\cos(3y) + 3xy^2}$$

4

0.1.8 Solution for Exam II

Question # 1 $\quad F(x,y,z) = \langle \frac{-1}{3}y, \frac{5}{9}x, zy \rangle$

$S: x^2+y^2+z^2 = 9$; Pink circle : $x^2+y^2+0^2 = 9$

$$\left.\begin{array}{l} x = 3\cos(t) \\ y = 3\sin(t) \end{array}\right\} \quad 0 \le t \le 2\pi$$

$\vec{R}(t) = \langle 3\cos(t), 3\sin(t), 0 \rangle$

$dR = \langle -3\sin(t), 3\cos(t), 0 \rangle \, dt$

$$\iint_S \text{curl} F \cdot dS = \int_{t=0}^{t=2\pi} \langle -\tfrac{1}{3}(3\sin t), \tfrac{5}{9}(3\cos t), 0 \rangle \cdot \langle -3\sin t, 3\cos(t), 0 \rangle \, dt$$

$$= \int_{t=0}^{t=2\pi} [3\sin^2 t + 5\cos^2 t] \, dt$$

$$= \int_{t=0}^{t=2\pi} [3(\sin^2 t + \cos^2 t) + 2\cos^2 t] \, dt$$

$$= \int_{t=0}^{t=2\pi} [3 + \tfrac{2}{2}(1+\cos 2t)] \, dt$$

$$= \int_{t=0}^{t=2\pi} [4 + \cos 2t] \, dt$$

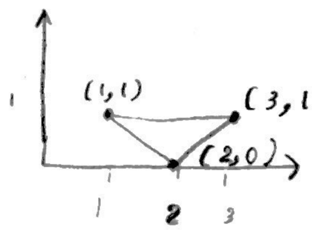
$$= 4t + \tfrac{1}{2}\sin 2t \Big|_0^{2\pi}$$

$= 8\pi$ unit

Question #2 $F(x,y,z) = \langle \frac{-1}{2}xz, yx, zy \rangle$

$S: z = x + 2y$

Region D: $(2,0), (3,1), (1,1)$



$$\int_C F \cdot dr = \iint_S \text{curl} F \cdot ds$$

$\text{curl} F = \langle F_{zy} - F_{yz}, F_{xz} - F_{zx}, F_{yx} - F_{xy} \rangle$

$= \langle z - 0, -\frac{1}{2}x - 0, y \rangle$

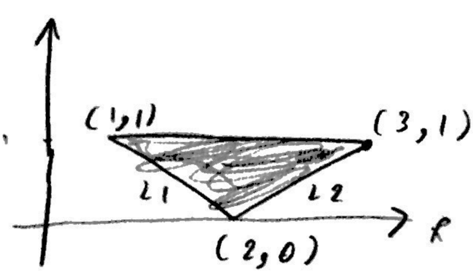
$= \langle z, -\frac{1}{2}x, y \rangle$

$z - x - 2y = 0 \implies g(x,y,z) = z - x - 2y$

$ds = \langle -g_x, -g_y, 1 \rangle$

$= \langle 1, 2, 1 \rangle$

$\iint_S \text{curl} f \cdot ds$



using region D

$dx\,dy$ is easier

$L_1: \; m = \dfrac{0-1}{2-1} = -1$

$y - 0 = -1(x-2)$

$y = -x + 2$

$\Rightarrow x = -y + 2$

$L_2: \; m = \dfrac{1-0}{3-2} = 1$

$y - 1 = 1(x-3)$

$y = x - 2 \Rightarrow x = y + 2$

$$\Rightarrow \int_{y=0}^{y=1} \int_{x=-y+2}^{x=y+2} \langle z, \tfrac{-1}{2}x, y \rangle \cdot \langle 1, 2, 1 \rangle \, dx \, dy$$

$$\Rightarrow \int_{y=0}^{y=1} \int_{x=-y+2}^{x=y+2} (z - x + y) \, dx \, dy$$

$$= \int_{y=0}^{y=1} \int_{x=-y+2}^{x=y+2} (x + 2y - x + y) \, dx \, dy$$

$$= \int_{y=0}^{y=1} \int_{x=-y+2}^{x=y+2} 3y \, dx \, dy = \int_{y=0}^{y=1} 3y \left[y+2-(-y+2) \right] dy$$

$$= \int_{y=0}^{y=1} 3y(2y) \, dy = \frac{6y^3}{3} \Big|_0^1 = 2$$

Question #3 $\quad \delta(x,y) = \sqrt{1+x+y}$

$z = \frac{2}{3}x^{3/2} + \frac{2}{3}y^{3/2}$ $\qquad$ D: $x^2+y^2 \le 4$, $x^2+y^2 \ge 1$

$$m = \iint_D \delta(x,y)\, dS$$

$$dS = \sqrt{1+(z_x)^2+(z_y)^2}\, dA$$

$dA = r\,dr\,d\theta$ $\qquad 1 \le r \le 2$, $\quad 0 \le \theta \le \pi/2$ $\qquad x = r\cos\theta$, $y = r\sin\theta$

$z_x = \frac{3}{2}\left(\frac{2}{3}\right)x^{1/2} = x^{1/2}$, $\quad z_y = \frac{3}{2}\left(\frac{2}{3}\right)y^{1/2} = y^{1/2}$

$$\therefore\ dS = \sqrt{1+x+y}\ dA$$

$$m = \iint_D \left[\sqrt{1+x+y}\right]\left[\sqrt{1+x+y}\right] dA$$

$$= \iint_D (1+x+y)\, dA \qquad x = r\cos\theta,\ y = r\sin\theta$$

$$= \int_{\theta=0}^{\theta=\pi/2}\int_{r=1}^{r=2} (1+r\cos\theta+r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_{\theta=0}^{\theta=\pi/2}\int_{r=1}^{r=2} \left[r + r^2(\cos\theta+\sin\theta)\right] dr\, d\theta$$

$\Rightarrow$

Continuation . Question # 3

$$= \int_{\theta=0}^{\theta=\pi/2} \left[\frac{1}{2} r^2 + \frac{1}{3} r^3 (\cos\theta + \sin\theta) \right]_{r=1}^{r=2} d\theta$$

$$= \int_0^{\pi/2} \left[\frac{1}{2}(2^2 - 1^2) + \frac{1}{3}(2^3 - 1^3)(\cos\theta + \sin\theta) \right] d\theta$$

$$= \int_{\theta=0}^{\theta=\pi/2} \frac{3}{2} + \frac{7}{3}(\cos\theta + \sin\theta) \ d\theta$$

$$= \frac{3}{2}\theta + \frac{7}{3}(\sin\theta - \cos\theta) \Big|_0^{\pi/2}$$

$$= \frac{3}{2} \pi/2 + \frac{7}{3}(2)$$

$$= \frac{3\pi}{4} + \frac{14}{3}$$

Question #4

check $f_{xy} = \sqrt{x(y-3)+1} + (y-3)\left(\frac{1}{2}\right)(x(y-3)+1)^{-1/2} x$

$= \sqrt{x(y-3)+1} + \dfrac{(y-3)x}{2\sqrt{x(y-3)+1}}$

$= \dfrac{x(y-3)+1+(y-3)x}{2\sqrt{x(y-3)+1}} = \dfrac{2x(y-3)+1}{2\sqrt{x(y-3)+1}}$

$f_{yx} = \sqrt{x(y-3)+1} + \dfrac{x(y-3)}{2\sqrt{x(y-3)+1}}$

$= \dfrac{x(y-3)+1+x(y-3)}{2\sqrt{x(y-3)+1}} = \dfrac{2x(y-3)+1}{2\sqrt{x(y-3)+1}}$

$\therefore$ conservative $(f_{xy} = f_{yx})$

$k(x,y)$

$k_x = (y-3)\sqrt{x(y-3)+1}$

$\int (y-3)\sqrt{x(y-3)+1}\,dx$

Let $u = x(y-3)+1$

$du = y-3\,dx$

$= \int \sqrt{u}\,du = \dfrac{2u^{3/2}}{3} = \frac{2}{3}[x(y-3)+1]^{3/2}$

$\Longrightarrow$

6

continuation Question #4

$\int F_Y \, dY = \int x \sqrt{x(Y-3)+1} \, dY$

let $u = x(Y-3)+1$

$du = x \, dy$

$\Rightarrow \int \sqrt{u} \, du = \frac{2u^{3/2}}{3} = \frac{2}{3}[x(Y-3)+1]^{3/2}$

$K(x,y) = \frac{2}{3}[x(Y-3)+1]^{3/2}$

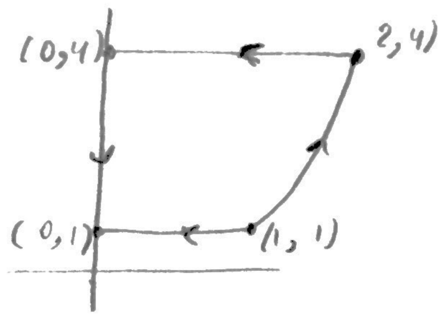
$\int_c \vec{F} \cdot dR = K(0, 3+e^{-3}) - K(3,4)$

$= \frac{2}{3}(1)^{3/2} - \frac{2}{3}[3(4-3)+1]^{3/2}$

$= \frac{2}{3} - \frac{2}{3}(4)^{3/2}$

$= \frac{-14}{3}$

Question #5 1



Region D

$0 \le x \le \sqrt{y}$

$1 \le y \le 4$

$$\int_C F \cdot dr = \iint_D (f_{yx} - f_{xy})\, dA$$

ccw (+ve direction)

$f_{yx} = \frac{3}{2}\sqrt{y^{3/2}+1}$, $f_{xy} = 0$

$$\int_C F \cdot dr = \iint_D \frac{3}{2}\sqrt{y^{3/2}+1}\, dA = \int_{y=1}^{y=4}\int_{x=0}^{x=\sqrt{y}} \frac{3}{2}\sqrt{y^{3/2}+1}\, dx\, dy$$

$$= \int_{y=1}^{y=4}\left[\left(\frac{3}{2}\sqrt{y^{3/2}+1}\;\; x\right)\Big|_0^{\sqrt{y}}\right] dy = \int_{y=1}^{y=4} \frac{3}{2}\sqrt{y}\,\sqrt{y^{3/2}+1}\, dy$$

Let $u = y^{3/2}+1$ $y=1$: $u=2$

$du = \frac{3}{2} y^{1/2}\, dy$ $y=2$: $u=9$

$$= \int_{u=2}^{u=9} \sqrt{u}\, du = \frac{2}{3} u^{3/2}\Big|_2^9$$

$$= 16.11$$

8

Question #6

i) curl F = $\langle F_{zy} - F_{yz},\ F_{xz} - F_{zx},\ F_{yx} - F_{xy} \rangle$

$= \langle 2z - 2z,\ 0-0,\ 0-0 \rangle = \langle 0, 0, 0 \rangle$

$\therefore$ yes it is conseVvative

ii) $K(x, y, z)$

$\int (x^2+1)\,dx = \frac{1}{3}x^3 + x$

$\int (z^2+1)\,dy = (z^2+1)y$

$\int (2zy+1)\,dz = z^2y + z$

$K(x,y,z) = \frac{1}{3}x^3 + x + z^2y + y + z$

$= \frac{1}{3}x^3 + z^2y + x + y + z$

checki

$K_x = x^2+1$

$K_y = z^2+1$

$K_z = 2zy+1$

$\int_c F \cdot dR = K(4,3,2) - K(1,1,1)$

$= [\frac{1}{3}(4)^3 + (2)^2(3) + 2 + 3 + 4] - [\frac{1}{3}(1)^3 + (1)^2(1) + 1 + 1 + 1]$

~~$= [\frac{64}{3} + 21] - [\frac{1}{3} + 4]$~~

$= [\frac{64}{3} + 21] - [\frac{1}{3} + 4] = 38$

9

0.1.9 Solution for Final Exam

Final Exam Calculus III

Question 1 :

$F(x,y,z) = \langle z^2, x^2, y^2 \rangle$

Use Stokes theorem

Question 5 :

$f(x,y) = \sqrt{x^2+y^2+4}$

(i) $D_u f(1,2)$ in direction of $\langle 3,4 \rangle$

$D_u f(x,y) = \nabla f(x,y) \cdot u$

$\nabla f = \langle f_x, f_y \rangle = \left\langle \frac{x}{\sqrt{x^2+y^2+4}}, \frac{y}{\sqrt{x^2+y^2+4}} \right\rangle$

$\nabla f(1,2) = \langle \frac{1}{3}, \frac{2}{3} \rangle$

$D_u f(1,2) = \nabla f(1,2) \cdot u$

(ii) Max rate of change at (1,2)

max $D_u f(x,y) = |\nabla f(x,y)|$ it happens in direction of $\nabla f(1,2)$

$\nabla f(1,2) = \langle \frac{1}{3}, \frac{2}{3} \rangle$

and max occurs in direction of $\nabla f(1,2)$

Question 2 :

(i) $\lim\limits_{(x,y)\to(0,0)} \frac{2e^{(2x-3y)} - y}{2x-3y}$ DNE

So, check with x-axis and y-axis

Thus, we can conclude that since we got diff answers, limit DNE

(ii)

Choosing diff path

$\lim\limits_{x\to 4} \frac{(-2x)^{\frac{1}{3}}-1}{2x+4} = \boxed{\frac{-1}{3}}$

Since Both Paths gave same answer and we know that lim exists so answer of limit is $\frac{-1}{3}$

$\boxed{\frac{-1}{2}}$

Question 8 :

(i) $f(x,y) = x^2+y^2+xy-3x-3y+20$

$f_x = 2x+y-3 = 0$] $x=1$
$f_y = 2y+x-3 = 0$] $y=1$ (1,1) critical point

$f_{xx} = 2$ $\quad f_{xy} = 1$
$f_{yy} = 2$

$D = f_{xx}(a,b) f_{yy}(a,b) - (f_{xy}(a,b))^2$ " at (1,1) "

$= 2 \times 2 - (1)^2 = \boxed{3} > 0$ and $f_{xx} > 0$ at (1,1)

$\therefore$ We have local min at (1,1)

(ii) $x, y, z = ?$ $\quad x,y,z > 0$ $\quad xyz$ max and $x+y+z^2 = 25$

$f(x,y,z)$ $\quad g(x,y,z)$

Lagrange

$\boxed{\nabla f = \lambda \nabla g}$

$\langle f_x, f_y \rangle = \lambda \langle g_x, g_y \rangle$

$\boxed{f_x = \lambda g_x}$

$\boxed{f_y = \lambda g_y}$

$\nabla g(x,y,z) = \langle 1, 1, 2z \rangle$

$\nabla f(x,y,z) = \langle yz, xz, xy \rangle$

$\nabla f = \lambda \nabla g$

$\langle yz, xz, xy \rangle = \lambda \langle 1, 1, 2z \rangle$

So

$yz = \lambda$
$xz = \lambda$] $y\cancel{z} = x\cancel{z}$ $\boxed{x=y}$
$xy = 2z\lambda$

$xy = 2z(xz)$

$y = 2z^2$

$z^2 = \frac{y}{2}$ $\Rightarrow z = \sqrt{\frac{y}{2}} = \sqrt{\frac{x}{2}}$

we have

$x + y + z^2 = 25$

$y + y + \frac{y}{2} = 25$

$\frac{5}{2} y = 25$

$\boxed{y = 10}$

$x = y$

$\boxed{x = 10}$

$z = \sqrt{\frac{y}{2}} = \sqrt{\frac{10}{2}}$

$\boxed{z = \sqrt{5}}$

So the three + numbers for x, y, z are 10, 10, $\sqrt{5}$ respectively

Question 6 :

1. $z > 0 \quad z^2 + ze^{(x-y)} - 6 = 0$

$x(t,s) = 3t^2 + 2s$

$y(t,s) = 2t + 3s$

$\frac{\partial z}{\partial t}$ when $(1,1)$ [t s]

differentiate both side with respect to r, t

Using Chain Rule : $2z \frac{\partial z}{\partial t} + \frac{\partial z}{\partial t} \cdot e^{(x-y)} + ze^{(x-y)} \frac{\partial}{\partial t}(x-y) - 0 = 0$

$\frac{\partial z}{\partial t}(2z + e^{(x-y)}) + ze^{(x-y)}\left[\frac{\partial x}{\partial t} - \frac{\partial y}{\partial t}\right] = 0$

$\quad + ze^{(x-y)}(3(2t) - 2) = 0$

$\frac{\partial z}{\partial t} = \frac{2e^{(x-y)}(6t-2)}{2z + e^{(x-y)}}$ when $t=1, s=1$ $\quad x = 3t^2+2s$, $x = 5$ $\quad y = 2t+3s$, $y = 5$

$e^{(x-y)} = e^0 = 1$ when $t=s=1$

$z^2 + ze^{(x-y)} - 6 = 0$

$z^2 + z - 6 = 0$

$z = 2, -3$

$e^{(x-y)} = 1$, $z = 2$

$\frac{\partial z}{\partial t} = \frac{2 \times 1 \times (6 \times 1 - 2)}{2 \times 2 + 1} = \frac{2 \times 1 \times 4}{4+1} = \boxed{\frac{8}{5}} = \left.\frac{\partial z}{\partial t}\right|_{t=1,s=1}$

2. $r(t) = \langle \sqrt{2t}, t^2+t, \sin(\pi t) \rangle$

Equation of tangent line at (2,6,0)

* $(2)^2 = (\sqrt{2t})^2$

$4 = 2t$

$t = 2$

$r'(t) = \langle \frac{1}{\sqrt{2t}}, 2t+1, \pi\cos(\pi t) \rangle$

$r'(2) = \langle \frac{1}{2}, 5, \pi \rangle$

Directional vector

* $6 = t^2 + t$

$t^2 + t - 6 = 0$

$t = 2$, $t = -3$ ✗

$x(t) = 2 + \frac{1}{2}t$

$y(t) = 6 + 5t$

$z(t) = \pi t$

* $0 = \sin(\pi t)$

$\boxed{t = 2}$

Equation of tangent line

$= \langle 2 + \frac{1}{2}t, 6+5t, \pi t \rangle$

$= \langle 2, 6, 0 \rangle + t \langle \frac{1}{2}, 5, \pi \rangle$

Question 3 :

$d(x,y) = x + y$

$z = 8 - 2x^2 - 2y^2 \Longrightarrow$ it's a circle cuz $8 = 2x^2 + 2y^2$

Mass = ? $\quad \div 2$ $\quad 4 = x^2 + y^2$

$\iint_D d(x,y)\, dS$

$dS = \sqrt{1 + f_x^2 + f_y^2} = \sqrt{1+4r^2}$

use polar : $\quad d(r,\theta) = r\cos\theta + r\sin\theta$

$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$dS = r\, dr\, d\theta$

Mass $= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (r\cos\theta + r\sin\theta) r\, dr\, d\theta$

$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} r^2(\cos\theta + \sin\theta)\, dr\, d\theta$

$\left. \frac{r^3}{3}(\cos\theta + \sin\theta) \right|_{r=0}^{r=2}$

$\int_{\theta=0}^{\theta=2\pi} \frac{8\sin\theta + \cos\theta}{3}\, d\theta$

$= \left. \frac{8\sin\theta - \cos\theta}{3} \right|_{\theta=0}^{\theta=2\pi}$

$= \boxed{0}$

Question 7 :

$F = \langle e^{(x-2z)} + y + z + 1, 2y + x + 1, -2e^{(x-2z)} + x + 3 \rangle$

(i) $\text{Curl}(F) = \langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy} \rangle$

$= \langle 0 - 0, (-2e^{x-2z} + 1) - (-2e^{x-2z} + 1), 1 - 1 \rangle$

$= \langle 0, 0, 0 \rangle$

$\therefore$ Yes F: conservative.

(ii) $\int_C F \cdot dr = k(6,8,3) - k(0,1,0)$ $\quad$ since Conservative then exist $k(x,y,z)$

$\int f_x\, dx = \int e^{x-2z} + y + z + 1\, dx = e^{x-2z} + yx + zx + x$

$\int f_y\, dy = \int 2y + x + 1\, dy = y^2 + xy + y$

$\int f_z\, dz = \int -2e^{(x-2z)} + x + 3\, dz = e^{x-2z} + xz + 3z$

$k(x,y,z) = e^{(x-2z)} + zx + x + yx + y^2 + y + 3z$

$\int_C F\, dr = k(6,8,3) - k(0,1,0)$

$= 154 - 3 = \boxed{151}$

Question 4 :

$F(x,y) = \langle e^{x^2+1}, xe^{(y^3+3y)} \rangle$

Green theorem $\iint_D f_{yx} - f_{xy}$

$f_{xy} = 0$

$f_{yx} = e^{y^3+3y}$

$W = \int_{y=0}^{y=2} \int_{x=0}^{x=y^2+1} [e^{y^3+3y} - 0]\, dx\, dy$

$\int_{x=0}^{x=y^2+1} e^{y^3+3y}\, dx = \left. xe^{y^3+3y} \right|_{x=0}^{x=y^2+1} = (y^2+1)e^{y^3+3y}$

$\int_{y=0}^{y=2} (y^2+1)e^{y^3+3y}\, dy = \boxed{400868}$

0.2 QUIZZES

0.2.1 Quiz I

Quiz I , MTH 203 , Fall 2021

Ayman Badawi

QUESTION 1. Convince me that $L_1 : x = 3t + 1, y = t - 2, z = -4t + 2$ ($t \in R$) is perpendicular to $L_2 : x = 2w, y = 2w - 5, z = 2w - 6$ ($w \in R$).

QUESTION 2. Is the line $L_1 : x = 3t + 1, y = t - 2, z = -4t + 2$ ($t \in R$) parallel to $L_2 : x = 6w - 5, y = 2w - 4, z = -8w + 11$ ($w \in R$)? EXPLAIN WHY YES or WHY NO.

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0.2.2 Quiz II

Name——————————————, ID ————————

Quiz II , MTH 203 , Fall 2021

Ayman Badawi

QUESTION 1. Does the line $L : x = -t + 2, y = 2t + 6, z = 3t - 2 (t \in R)$ intersect the plane $P : x + y + 2z = 25$. If yes, then find the point of intersection.

QUESTION 2. Can we draw the vector $v =< 2, -4, -3 >$ inside the plane $P : x + y - 2z = 12$? explain

QUESTION 3. Does the plane $P_1 : x + 2y + z = 5$ intersect the plane $P_2 : -x - y + 3z = -2$? If yes, the find the parametric equations of line of intersection.

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0.2.3 Quiz III

Name————————————————, ID ————————

Quiz III , MTH 203 , Fall 2021

Ayman Badawi

QUESTION 1. Find all critical points of the function $f(x, y) = yx^2 - 2x^2 - y^2$. Then classify each critical point as local Max/Min or Saddle or neither.

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0.2.4 Quiz IV

Quiz IV , MTH 203 , Fall 2021

Ayman Badawi

QUESTION 1. Let $f(x, y) = ye^{(x-4)} + x\sqrt{y} + 2x$.

(i) Let $a = f(4, 1)$. Find a

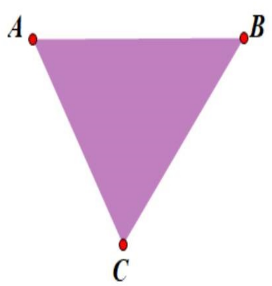
(ii) Find f_x and f_y.

(iii) Let P be the tangent plane to $f(x, y)$ at the point $(4, 1, a)$. Let N be a vector that is perpendicular to P. Find N.

(iv) Find the equation of P, where P is as in (iii).

(v) Use the concept of the tangent plane to approximate $f(4.2, 0.8)$

QUESTION 2. A solid object has a a triangular base that is bounded by $y = x$ and $y = -x$ (see PICTURE). Note that $-2 \le x \le 2$ and $0 \le y \le 2$. The height is determined by the function $f(x, y) = e^{\sqrt{4-y^2}}$. Find the volume of such object.



A solid object has a triangular base in the xy-plane as in picture. Note A = (-2, 2), B = (2, 2), and C = (0, 0). C, B lie on the line y = x, C and A lie on y = -x.

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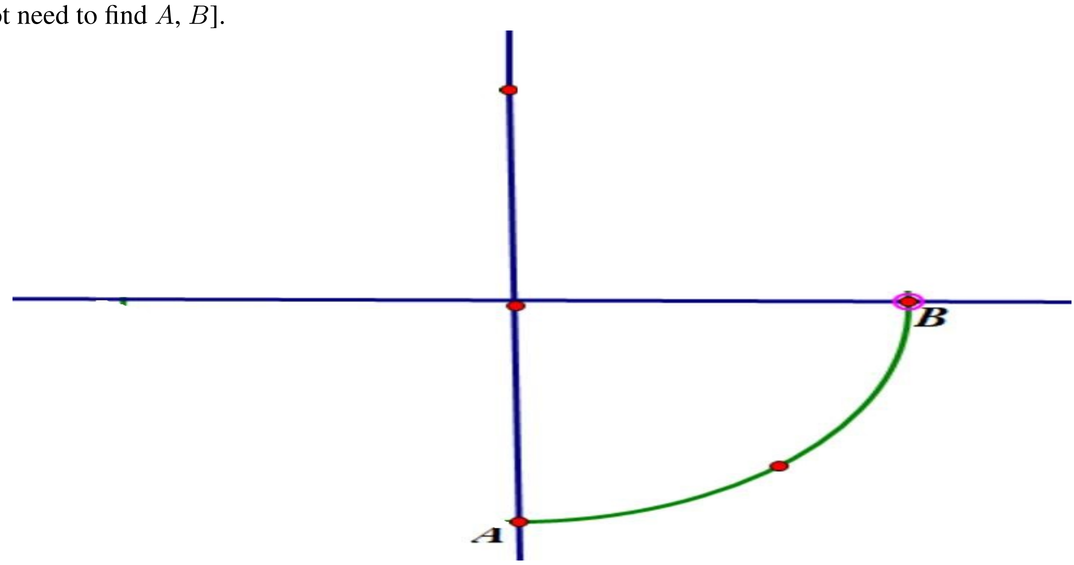
0.2.5 Quiz V

Quiz V , MTH 203 , Fall 2021

Ayman Badawi

SHOW THE WORK, SUBMIT by 2:35pm

QUESTION 1. See the below picture. A force $F(x, y) =< -y, x >$ is acting on a particle in order to move it from the point A to the point B along the ellipse $x^2 + 4y^2 = 16$. Find the work done by the force $F(x, y)$. [Hint: you do not need to find A, B].



QUESTION 2. The height of a curtain is determine by $f(x, y) = e^{2x} + 5y^2$ defined over the curve $y = e^x$, $0 \leq x \leq ln(5)$. Find the surface area of the curtain.

QUESTION 3. Evaluate the integral $\int_C (1 + 2y)dy$, where the curve C is $y = e^{x^2}$, $0 \leq x \leq 1$

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0.2.6 Quiz VI

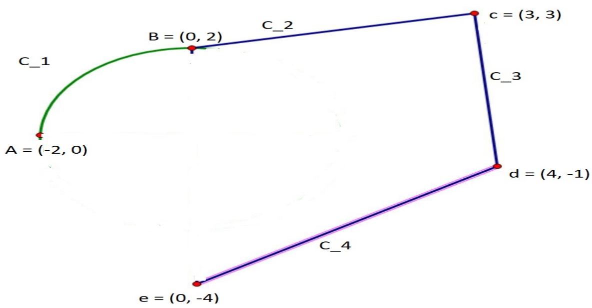
Name——————————————, ID ——————

Quiz VI, MTH 203 , Fall 2021

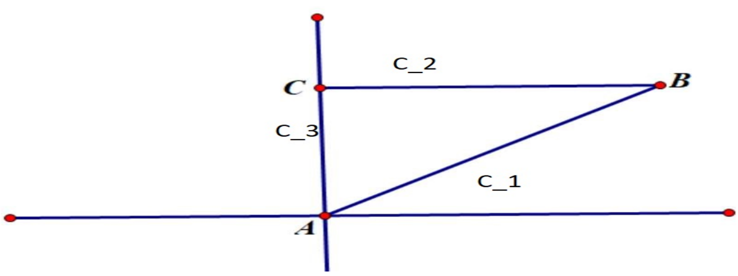
Ayman Badawi

SHOW THE WORK, SUBMIT by 2:35pm

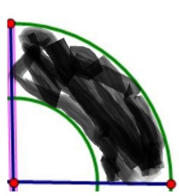
QUESTION 1. See the below picture. A force $F(x,y) = < ye^{yx} - 2x, xe^{yx} + 4y^3 >$ is acting on a particle in order to move it from the point $A = (-2, 0)$ to the point $e = (0, -4)$ along the curve C that consists of C_1 part of the circle $x^2 + y^2 = 4$, C_2 the line segment between B and c, C_3 the line segment between c and d, and C_4 the line segment between d and e. Find the work done by the force $F(x,y)$.



QUESTION 2. See the below picture. A force $F(x,y) = < -2y,\ \frac{2}{3}x\sqrt{y^2+9} >$ is acting on a particle in order to move it from the point $A = (0.0)$ then back to the same point A along the curve C that consists of C_1 part of the the line $y = 0.5x$ between A and B, C_2 part of the line $y = 4$ between B and C, and C_3 part of the y-axis between C and A. Use Green's Theorem to find the work done by the force $F(x,y)$. [Hint: Is dxdy or dydx easier?]



QUESTION 3. Find the surface area of the part of $f(x,y) = \frac{x^2}{2\sqrt{2}} + \frac{y^2}{2\sqrt{2}}$ defined over the region in the first quadrant of the xy-plane bounded by $x^2 + y^2 \le 4$ and $x^2 + y^2 \ge 1$, see picture.



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0.3 Exams

0.3.1 Exam I

Name———————————, ID ————————

Exam I , MTH 203 , Fall 2021

Ayman Badawi

(IMPORTANT: STOP WORKING at 7:40 pm, Go to assessment, you will find a folder SUBMIT EXAM ONE, Submit your solution as a PDF file, Max by 7:55 pm)

Score = $\frac{}{62}$

QUESTION 1. (6 points) Given $Z = f(x,y) = \frac{36y}{9x^4+1}$ is defined over the arc $C : y = x^3$, where $0 \leq x \leq 1$. Find the surface area of the curtain that is determined by $f(x,y)$ and the ARC C. [Hint: Note that $Z = f(x,y) \geq 0$]. Show all steps that you used in evaluating the integral. You may use a calculator at the end (i.e., last step ONLY, in order to come up with the answer)

QUESTION 2. (6 points, SHOW THE WORK) Given $Z = f(x,y) = x\sqrt{y+1}$ is defined over the region bounded by the positive x-axis, $y = x$, and $0 \leq x \leq 1$. Find the volume of such object. [Hint: Note that $Z = f(x,y) \geq 0$]. Show all steps that you used in evaluating the integral. You may use a calculator at the end (i.e., last step ONLY, in order to come up with the answer)

QUESTION 3. (10 points, SHOW THE WORK) Given $Z = f(x,y) = 5x - 8y + 2xy - x^2 + \frac{1}{3}y^3$

(i) Find all critical points of $f(x,y)$.

(ii) Classify each critical point as local max., local min., saddle point, or neither.

QUESTION 4. (8 points, SHOW THE WORK) Find the absolute maximum and the absolute minimum of $f(x,y) = x^2 + 12x - y^2 + 6$ over the bounded region $x^2 + y^2 \leq 25$. [Hint: the region consists of all points inside the circle $x^2 + y^2 = 25$ including the points on the circle $x^2 + y^2 = 25$].

QUESTION 5. (6 points, SHOW THE WORK) Given $P_1 : 2x + 4y + 5z = 6$ and $P_2 : 6x + ay + bz = c$ such that P_1 is parallel to P_2. Find all possible values of a, b, c.

QUESTION 6. (8 points, SHOW THE WORK) The plane $4x+11y-5z = -3$ intersects the plane $-2x-5y-z = 3$ in a line L. Find a parametric equations of L.

QUESTION 7. (i) **(2 points)** Is $4x^2 + 9y^2 + z^2 = 25$ a sphere or ellipsoid or a cone? Explain briefly

(ii) **(2 points)** Is $x^2 + y^2 = 9$ a cylinder of finite height? or a sphere of infinite radius? or neither? explain briefly

(iii) **(2 points)** Given $(3, -5, 6)$ is a local maximal point of $f(x,y)$. Find the equation of the tangent plane at $(3, -5, 6)$[Hint: pause, think! trust me it is not difficult]

(iv) Let $f(x,y) = 3xy + x^2 - \sqrt{y} + x^3y$. Let $a = f(1,4)$.

(1) Find a. **(1 point)**

(2) **(5 points)** Find the equation of the tangent plane to $f(x,y)$ at $(1,4,a)$.

(3)**(2 points)** Use the concept of the tangent plane to approximate $f(0.8, 4.01)$

QUESTION 8. (4 points, SHOW THE WORK) Use the concept of partial order to find dy/dx, where $y^2e^{5x} + xsin(3y) + x^2 + xy^3 + 10 = 0$

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0.3.2 Exam II

Name———————————————, ID ———————

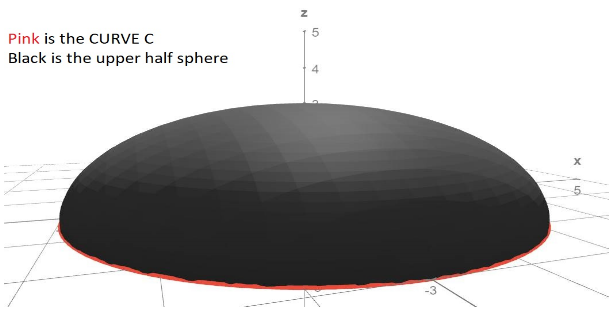
Exam II , MTH 203 , Fall 2021

Ayman Badawi

(IMPORTANT: STOP WORKING at 10:40 pm, Go to assessment, you will find a folder SUBMIT EXAM TWO, Submit your solution as a PDF file,by 10:55 pm, as at most. I will not receive solutions by EMAIL)

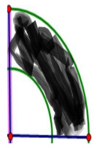
Score = $\frac{\qquad\qquad}{36}$

QUESTION 1. (**6 points, SHOW THE WORK**) Given $F(x, y, z) =< \frac{-1}{3}y, \frac{5}{9}x, zy >$ defined over the upper-half of the sphere $S : x^2 + y^2 + z^2 = 9$ (i.e., $0 \leq z \leq 3$). Assume that S is oriented upward. Use Stoke's Theorem to evaluate $\int\int_S Curl(F) \cdot dS$. [Hint: Find the (familiar) curve C that is surrounding the upper half sphere!]



QUESTION 2. (**6 points, SHOW THE WORK**) Given $F(x, y, z) =< \frac{-1}{2}xz, yx, zy >$ defined over the portion of the plane $S : z = x + 2y$ oriented upward that is bounded by the triangular curve C positively oriented (i.e., ccw) with vertices $(2, 0, 2), (3, 1, 5), (1, 1, 3)$. Use Stoke's Theorem to find $\int_C F \cdot dr$. [hint: in order to find the region D in the xy-plane, project the vertices of the triangle over the xy-plane (i.e., let $z = 0$), then stare at the region D inside the triangle]

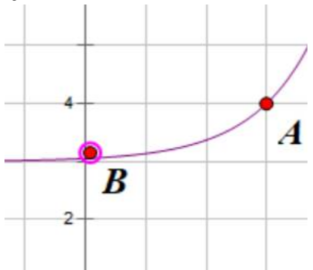
QUESTION 3. (**6 points, SHOW THE WORK**) The density function of an object is given by $d(x, y) = \sqrt{1 + x + y}$. The surface of th object has the shape that is determined by $z = \frac{2}{3}x^{\frac{3}{2}} + \frac{2}{3}y^{\frac{3}{2}}$ defined over the region D (see picture below) in the first quadrant of the xy-plane where $x^2 + y^2 \leq 4$ and $x^2 + y^2 \geq 1$. Find the mass of such object. [Hint: Note that the mass is $\int\int_D d(x, y)dS$]



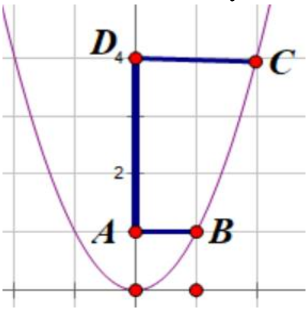
QUESTION 4. (**6 points, SHOW THE WORK**) See the below picture. A force

$$F(x, y) =< (y - 3)\sqrt{x(y - 3) + 1},\ x\sqrt{x(y - 3) + 1} >$$

is acting on a particle in order to move it from the point $A = (3, 4)$ to the point $B = (0, 3 + e^{-3})$ along the curve $C : y = 3 + e^{(x-3)}$ (clockwise). Find the work done by the force $F(x, y)$.



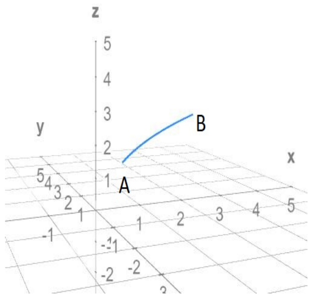
QUESTION 5. (**6 points, SHOW THE WORK**) See the below picture. A force $F(x,y) =< x^2+x+1,\ \frac{3}{2}x\sqrt{y^{\frac{3}{2}}+1} >$ is acting on a particle in order to move it from the point $A = (0,1)$ then back to the point A along the curve C (counter clockwise) that consists of C_1: part of the line $y = 1$ from A to $B = (1,1)$, C_2: part of the curve $y = x^2$ from B to $C = (2,4)$, C_3: part of $y = 4$ from C to $D = (0,4)$, and C_4: part of the y-axis from D to A. Use Green's Theorem to Find the work done by the force $F(x,y)$.



QUESTION 6. (**6 points, SHOW THE WORK**) Let $F =< x^2+1,\ z^2+1,\ 2zy+1 >$

(i) Find Curl(F). Is F conservative?

(ii) Assume that the given F is a force that is acting on a particle in order to move it from $A = (1,1,1)$ to the point $B = (4,3,2)$ along the curve $r(t) =< t,\ 2\sqrt{t}-1,\ \sqrt{t} >$. Find the work done by F. See the picture of $r(t)$ below



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0.3.3 Final Exam

Name——————————————, **ID** ————————

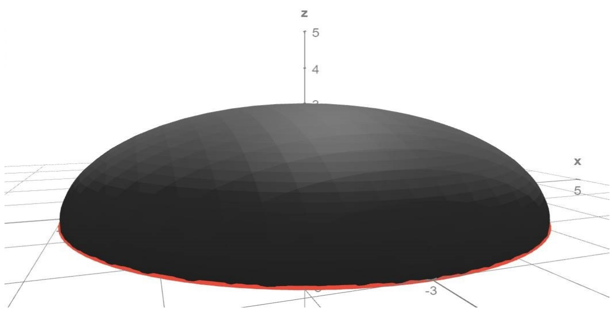
Final Exam , MTH 203 , Fall 2021

Ayman Badawi

(IMPORTANT: STOP WORKING at 4:00 pm, Go to assessment, you will find a folder SUBMIT Final Exam, Submit your solution as a PDF file,by 04:12 pm, as at most. I will not receive solutions by EMAIL)

Score = $\frac{\qquad\qquad}{50}$

QUESTION 1. (6 points, SHOW THE WORK) Given $F(x, y, z) =< y^2x, x^2y, \frac{1}{3}z^3 >$ defined over the solid upper-half of the sphere $S : x^2 + y^2 + z^2 = 4$ (i.e., $0 \le z \le 2$). Assume that S is oriented upward and closed from the bottom by the plane $z = 0$. Use the Divergence's Theorem to find the flux through the given solid upper half sphere (i.e., $\int\int_S F(x, y, z) \cdot dS$).

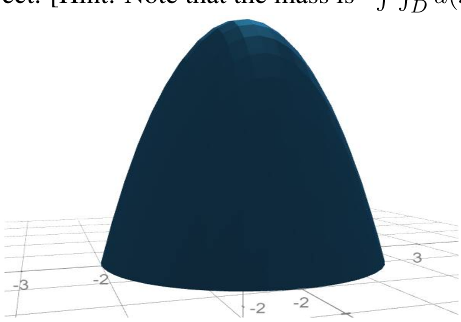


QUESTION 2. (6 points, SHOW THE WORK)

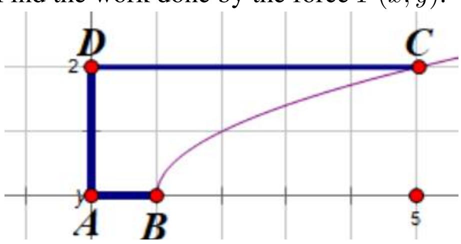
(i) Show that $Lim_{(x,y)\to(5,2)} \dfrac{2e^{(2x-5y)}-y}{2x-5y}$ does not exist.

(ii) Given $Lim_{(x,y)\to(4,3)} \dfrac{\sqrt[3]{3y-2x} \;\; -1}{2x-3y+1}$ exists. What is it?

QUESTION 3. (6 points, SHOW THE WORK) The density function of an object is given by $d(x, y) = x + y$. The surface of th object is determined by $z = 8 - 2x^2 - 2y^2$ above the xy-plane (see picture). Find the mass of such object. [Hint: Note that the mass is $\int\int_D d(x, y)\, dS$]



QUESTION 4. (6 points, SHOW THE WORK) See the below picture. A force $F(x, y) =< e^{(x^2+1)}, xe^{(y^3+3y)} >$ is acting on a particle in order to move it from the point $A = (0, 0)$ then back to the point A along the curve C (counter clockwise) that consists of C_1: part of the x-axis from A to $B = (1, 0)$, C_2: part of the curve $y = \sqrt{x-1}$ from B to $C = (5, 2)$, C_3: part of $y = 2$ from C to $D = (0, 2)$, and C_4: part of the y-axis from D to A. Use Green's Theorem to Find the work done by the force $F(x, y)$.



QUESTION 5. (6 points, SHOW THE WORK) Given $f(x, y) = \sqrt{x^2 + y^2 + 4}$.

(i) Find $D_u(1, 2)$, where u is the unit vector in the direction of $< 3, 4 >$.

(ii) Find the maximum rate of change of $f(x, y)$ at $(1, 2)$ and the direction in which the maximum rate of change occurs.

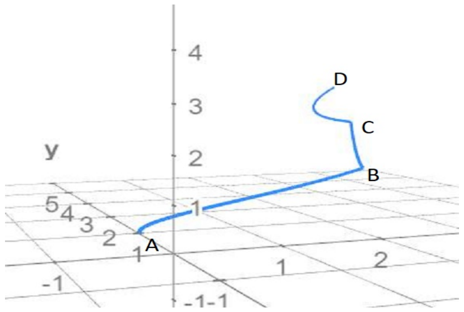
QUESTION 6. (6 points, SHOW THE WORK) (1) Given $z > 0$ and $z^2 + ze^{(x-y)} - 6 = 0$, $x(t, s) = 3t^2 + 2s$ and $y(t, s) = 2t + 3s$. Find $\partial z/\partial t$ when $t = s = 1$.

(2) Given the curve $r(t) =< \sqrt{2t},\ t^2 + t,\ sin(\pi t) >$. Find the equation of the tangent line to the curve $r(t)$ at $(2, 6, 0)$.

QUESTION 7. (6 points, SHOW THE WORK) Let $F =< e^{(x-2z)} + y + z + 1,\ 2y + x + 1,\ -2e^{(x-2z)} + x + 3 >$

(i) Find Curl(F). Is F conservative?

(ii) Assume that the given F is a force that is acting on a particle in order to move it from $A = (0, 1, 0)$ to the point $D = (6, 8, 3)$ along the curve C (see picture) that consists of C_1: part of $r(t) =< 3t, t^2 + 1, \sqrt{t} >$ from A to $B = (3, 2, 1)$, C_2: part of the curve $r(t) =< 3 + t, 2t + 2, t^2 + 1 >$ from B to $C = (4, 4, 2)$, C_3: part of $r(t) =< 2t^2 + 4, 4t + 4, t^2 + 2 >$ from C to $D = (6, 8, 3)$. Find the work done by the force $F(x, y)$ (i.e., find $\int_C F(x, y, z) \cdot dr$).



QUESTION 8. (8 points, SHOW THE WORK)

(i) Let $f(x, y) = x^2 + y^2 + xy - 3x - 3y + 20$. Find all critical points of $f(x, y)$ and classify each point as local min, local max, or saddle point

(ii) Find three positive real numbers x, y, z (i.e., $x, y, z > 0$) such that xyz is maximum and $x + y + z^2 = 25$. [hint: use Lagrange]

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